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Effects of rotation and tidal distortions on the equilibrium structure and periods of oscillations of stars in binary systems

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Abstract. We present methods for computing the equilibrium structures and periods of barotropic modes of oscillations of stars which are the primary components of binary systems. The methods utilize the averaging technique of Kippenhahn & Thomas and the concepts of Roche equipotentials to incorporate rotational and tidal effects. The use of the methods in obtaining the equilibrium structures and eigenfrequencies of barotropic modes of oscillations of certain polytropic models of stars (assumed to be the primary components of synchronous as well as non-synchronous binary systems) is also illustrated

Key words : binary stars--oscillations—rotation—tidal distortions

Introduction

The equilibrium structure of a star in a binary system is not perfectly spherically symmetric. The rotational forces as well as the tidal forces caused by the gravitational pull of the companion star, distort its otherwise spherically symmetric configuration. Some of the stars in binary systems such as 16 Lac (Fitch 1969) are also known to be pulsating stars. Pulsation periods of such stars in binary systems are also influenced by the effects of rotational and tidal forces.

The problem of determining the equilibrium structure of a star under the combined influence of rotational and tidal forces and determining its eigen frequencies of oscillations is mathematically quite a complex problem. Approximate methods have generally been applied in which the effect of only one of these two types of distortional forces is considered. Kopal (1972, 1974) developed in detail the concept of Roche equipotentials to study the binary systems. Kippenhahn & Thomas (1970) proposed a method for determining the equilibrium structures of rotationally and tidally distorted stars. Mohan & Saxena (1983) used the Kippenhahn & Thomas approach in conjunction with certain results on Roche equipotentials as obtained by Kopal (1972, 1974) and Mohan & Singh (1982) to obtain the equilibrium structures of rotationally and/or tidally distorted polytropic models of the stars. They also used this approach to determine the eigen frequencies of radial and nonradial modes of oscillations of rotationally and (or)

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tidally distorted polytropic models of the stars by assuming such oscillations to be barotropic (Mohan & Saxena 1985). Later Mohan & Agarwal (1987) also used this approach to determine the equilibrium structures and periods of oscillations of certain rotationally and/or tidally distorted composite models of the stars.

In this paper we present the method based on this approach for determining the equilibrium structures and periods of barotropic modes of oscillations of the primary components of stars in binary systems. The use of this technique in determining the equilibrium structure and periods of pulsations of certain stars, assumed to be primary components of binary systems, is also illustrated.

2. Kippenhahn and Thomas approach for determining the equilibrium structure of rotationally and tidally distorted stellar models

In order to study the effects of rotation and tidal distortions on the equilibrium structure of a gaseous sphere Kippenhahn & Thomas (1970) define certain topologically equivalent spherical surfaces to correspond to the actual equipotential surface of a rotationally and tidally distorted stellar model. They define on these equivalent spherical surfaces quantities such as $\bar{f}$, $\bar{g}$, etc. which denote certain averages of the quantities f, g respectively on the actual equipotential surfaces. If ψ denotes the total potential (gravitational, rotational, and tidal) of a rotationally and tidally distorted model at an arbitrary point (x, y, z) then $\psi\,(x, y, z) =$ const. is an equipotential surface. Let V_ψ be the volume enclosed and S_ψ the surface area of this equipotential surface $\psi =$ constant. Then in analogy with a sphere, Kippenhahn & Thomas define a variable r_ψ by the relation

$$V_\psi = \frac{4}{3}\pi r_\psi^3 \qquad \ldots(1)$$

r_ψ thus defined is then used to denote the radius of the topologically equivalent spherical surface. They also define the mean value of any function $f(x, y, z)$ over the equipotential surface $\psi =$ constant by the relation

$$\bar{f} = \frac{1}{S_\psi}\int_{\psi = \text{const.}} f\,d\sigma, \qquad \ldots(2)$$

where $d\sigma$ denotes the surface element of the equipotential surface $\psi =$ constant. $\bar{f}$ thus defined over the topologically equivalent spherical surface is used to represent the actual value of f over the equipotential surface $\psi =$ constant. Clearly $\bar{f}$ is a function of ψ only and can be determined for each equipotential surface $\psi =$ constant. Also by definition

$$S_\psi = \int_{\psi = \text{const.}} d\sigma. \qquad \ldots(3)$$

Corresponding to the usual definition of g, the acceleration due to gravity, Kippenhahn & Thomas define a function $g(x, y, z)$ by the relation

$$g = \frac{d\psi}{dn}, \qquad \ldots(4)$$

dn being the distance between two neighbouring equipotential surfaces $\psi =$ constant and $\psi + d\psi =$ constant. The distance dn in general is not constant. Thus using equation (2) the mean values of g and g^{-1} respectively are defined by the relations

$$\bar{g} = \frac{1}{S_\psi}\int_{\psi=\text{const}} \frac{d\psi}{dn}\,d\sigma\ ;\ \text{and}\ g^{-1} = \frac{1}{S_\psi}\int_{\psi=\text{const}} \left(\frac{d\psi}{dn}\right)^{-1} d\sigma. \qquad \ldots(5)$$

The mean values $\bar{g}$ and $\bar{g}^{-1}$ defined over the equipotential surface ψ = constant are clearly functions of ψ only and represent the value of g, g^{-1} over the topologically equivalent spherical surface.

The volume dV_ψ enclosed between the equipotential surfaces ψ = constant and $\psi + d\psi$ = constant is obtained as

$$dV_\psi = \int_{\psi=\text{const}} dn\ d\sigma = d\psi \int_{\psi=\text{const}} (d\psi)^{-1}\,d\sigma = S_\psi g^{-1} d\psi. \qquad \ldots(6)$$

Kippenhahn & Thomas (1970) also define non-dimensional parameters u, v, w as

$$u = \frac{S_\psi}{4\pi r_\psi^2},\quad v = \bar{g}\,\frac{r_\psi^2}{M_\psi},\quad \text{and}\ w = \bar{g}^{-1}\,\frac{GM_\psi}{r_\psi^2}, \qquad \ldots(7)$$

where M_ψ is the mass enclosed by the equipotential surface ψ = constant.

Thus the distorted equipotential surface ψ = constant is regarded as topologically equivalent to a spherical surface of radius r_ψ for which the various quantities are defined by the above relations. (It may be noted that if ψ is the gravitational potential of a sphere then ψ = constant are spherical surfaces with $r_\psi = r$ and therefore, $u = 1$. Also on these spherical surfaces $g = GM_\psi/r_\psi^2$ is constant and therefore v and w are also constant and equal to one).

Equations (1)-(7) are purely mathematical definitions which have been applied by Kippenhahn & Thomas to gravitational fields of gaseous sphere distorted by rotation and tidal forces. In hydrostatic equilibrium the equipotential surfaces are also surfaces of equipressure and equidensity. Therefore on an equipotential surface the pressure P_ψ, and the density ρ_ψ are also constant. Utilizing these concepts, Kippenhahn & Thomas obtain the equations governing the equilibrium structure of a rotationally and tidally distorted stellar model as

$$\frac{dM_\psi}{dr_\psi} = 4\pi r_\psi^2\,\rho_\psi,\quad \frac{dP_\psi}{dM_\psi} = -\frac{GM_\psi}{4\pi r_\psi^4} f_P,$$

$$\text{and}\ \frac{dL_\psi}{dM_\psi} = \epsilon,\quad \frac{dT_\psi}{dM_\psi} = -\frac{3\,KL_\psi}{64\pi^2 a\,C\,T_\psi^3 r_\psi^4} f_T, \qquad \ldots(8)$$

where

$$f_P = \frac{1}{uw}\ \text{and}\ f_T = \frac{1}{u^2 vw}.$$

In these equations L_ψ denotes the energy which passes per second through the equipotential surface ψ = constant, T_ψ is the temperature on this equipotential surface, and ϵ the rate of energy generation. Other symbols have usual meanings.

The boundary conditions to be satisfied are:

$$M_\psi = 0,\ L_\psi = 0\ \text{at centre}\ r_\psi = 0$$

and on the free surface $r_\psi = r_{\psi s}$

$$M_\psi = M_0,\ L_\psi = L_{\psi s},\ P_\psi = 0\,(\text{or}\ P_{\psi s}),\ T_\psi = 0\ (\text{or}\ T_{\psi s}) \qquad \ldots(9)$$

here M_0 is the total mass of the model and $L_{\psi s}$, $P_{\psi s}$, $T_{\psi s}$ are the values of L_ψ, P_ψ, T_ψ respectively on the outermost equipotential surface.

In the case of no distortion $f_p = f_T = 1$ and the above equations reduce to the usual equations governing the equilibrium structure of an undistorted gaseous sphere. In order to determine the inner structure of the distorted gaseous sphere the system of equations (8) has to be integrated numerically subject to the boundary conditions (9).

3. The proposed method for determining the equilibrium structure of the primary component of a star in a binary system

The evaluation of the actual equipotential surfaces of a rotationally and tidally distorted gas sphere is quite a complicated problem. Keeping this in view, Kippenhahn & Thomas (1970) proposed that for the evaluation of the parameters u, v, w, f_p, f_T etc., the actual equipotential surfaces of the distorted star could be replaced by appropriate Roche equipotential surfaces. (It may be noted that this approximation is reasonably valid for most of the real stars. In fact as far back as 1933, Chandrasekhar has shown that for stars whose central density bears to the mean density a ratio of 100 or more, the Roche model of a rotating configuration will represent the actual form of the surfaces of the star within an error of less than 1%).

A binary system of stars consists of a pair of stars rotating about their axes as well as revolving around their common centre of mass. Because of rotation and tidal effects of the companion, the components of binary system become rotationally and tidally distorted. In order to investigate the problems of structure and stability of such binary stars, the concepts of Roche equipotentials and Roche limits have often been used in literature. In such a system the primary star is supposed to be much more massive than the companion secondary star which is considered to be a point mass. The structure of the primary star is taken to be a Roche model in which the whole mass of the star is concentrated at its centre which is surrounded by an evanescent envelope in which density varies inversely as the square of distance from the centre. Some of the important results on Roche equipotentials relevant to the present study have been obtained by Kopal (1972) and Mohan & Singh (1982). Using these results, the system of differential equations (8) governing the equilibrium structure of the primary component of a star in binary system may be expressed as

$$\frac{dM_\psi}{dr_0} = 4\pi D^3 \rho_\psi r_0^2 f_1,$$

$$\frac{dP_\psi}{dr_0} = -\frac{GM_\psi}{Dr_0^2}\rho_\psi f_2,$$

$$\frac{dL_\psi}{dr_0} = 4\pi\epsilon D^3 \rho_\psi r_0^2 f_1,$$

$$\text{and } \frac{dT_\psi}{dr_0} = -\frac{3KL_\psi}{16\pi^2 \mathrm{Dac}\, T_\psi^3}\cdot\frac{\rho_\psi}{r_0^2} f_3 \text{ (radiative)},$$

$$= \left(1-\frac{1}{\gamma}\right)\frac{T_\psi}{\rho_\psi}\frac{d\rho_\psi}{dr_0} \text{ (convective)}. \quad \ldots(10)$$

Here $r_0 = \frac{1}{\psi^* - q}$;

ψ^* being total potential (both rotational and tidal) in non-dimensional form at a distance r/D from the centre of primary component, D being the distance between the centres of two components, q is the ratio of the mass of the secondary to that of the priminary. Also f_1, f_2, f_3 are certain functions of distortion parameters q, r_0 and $n(n = \omega^2/2$, ω^2 being the square of the non-dimensional angular velocity of rotation of the primary) and incorporate the effects of rotation and tidal distortions on the equilibirium structure equations of a stellar model.

Expressed explicitly in terms of n, q and r_0 these distortion parameters are

$$f_1 = 1 + 4nr_0^3 + \left(\frac{36}{5}q^2 + \frac{72}{15}nq + \frac{864}{45}n^2\right)r_0^6 + \frac{55}{7}q^2 r_0^8 + \frac{26}{3}q^2 r_0^{10} - \ldots,$$

$$f_2 = 1 - \left(\frac{2}{5}q^2 + \frac{4}{15}nq + \frac{48}{45}n^2\right)r_0^6 - \frac{9}{14}q^2 r_0^8 - \frac{8}{9}q^2 r_0^{10} - \ldots,$$

$$f_3 = 1 + \frac{4n}{3}r_0^3 + \left(\frac{6}{5}q^2 + \frac{12}{15}nq + \frac{224}{45}n^2\right)r_0^6 + \frac{24}{14}q^2 r_0^8 + \frac{20}{9}q^2 r_0^{10} + \ldots, \quad \ldots(11)$$

with

$$r_0 = r_\psi^* \left[1 - \frac{2n}{3}r_\psi^{*3} - \left(\frac{4}{5}q^2 + \frac{8}{15}nq - \frac{4}{45}n^2\right)r^{*6} - \frac{5}{7}q^2 r_\psi^{*8} - \frac{2}{3}q^2 r_\psi^{*10} - \ldots\right].$$

In the above expressions, terms up to second order of smallness in distortion parameters n and q are retained. The boundary conditions are again same as equations (9).

At the free surface, $r_0 = r_{0s}$ where

$$r_{0s} = \frac{1}{\psi_s^* - q}, \quad \ldots(12)$$

ψ_s^* being the nondimensional value of the total potential ψ on the outermost equipotential surface of the rotationally and tidally distorted stellar model.

It may be noted that by approximating the equipotential surfaces of a rotationally and tidally distorted model by Roche equipotentials, the structure of the star is not approximated by the structure of a roche model. This is evident from the fact that in case of no distortion ($n = q = 0$) $f_1 = f_2 = f_3 = 1$ and the system of differential equations (11) reduces to equations governing the equilibrium structure of the original undistorted star and not of the undistorted Roche model.

Usual numerical methods for solving the stellar structure equations can be used to integrate the system of differential equations (10). However, at each step, the values of the distortion parameters f_1, f_2, f_3 must be obtained using condition (11). In case the star under consideration is the primary component of a synchronous binary system we should also set $n = (q + 1)/2$.

The shapes of the various equipotentials of the distorted star, the volume V_ψ enclosed within various equipotential surfaces, and the surface areas S_ψ of these equipotential surfaces can be obtained using equations (13)-(15):

$$r_\psi = Dr_0 \left[1 + \frac{2n}{3} r_0^3 + \left(\frac{4}{5} q^2 + \frac{8}{15} nq + \frac{76}{45} n^2 \right) r_0^6 + \frac{5}{7} q^2 r_0^8 + \frac{2}{3} q^2 r_0^{10} + \ldots \right], \qquad \ldots(13)$$

$$V_\psi = \frac{4}{3} \pi D^3 r_0^3 \left[1 + 2n r_0^3 + \left(\frac{12}{5} q^2 + \frac{8}{5} nq + \frac{32}{5} n^2 \right) r_0^6 + \frac{15}{7} q^2 r_0^8 + 2 q^2 r_0^{10} + \ldots \right], \qquad \ldots(14)$$

$$S_\psi = 4\pi D^2 r_0^2 r_\psi = Dr_0 \left[1 + \frac{4n}{3} r_0^3 + \left(\frac{7}{5} q^2 + \frac{14}{15} nq + \frac{56}{15} n^2 \right) r_0^6 + \frac{9}{7} q^2 r_0^8 + \frac{11}{9} q^2 r_0^{10} + \ldots \right], \qquad \ldots(15)$$

4. Proposed method for computing the eigenfrequencies of certain barotropic modes of oscillation

A distorted model in a binary system is capable of performing small periodic adiabatic oscillations in a variety of modes of which some will be barotropic and other baroclinical. Whereas in the case of barotropic modes the fluid elements on an equipotential surface oscillate in unision and always remain so during oscillations, it is not so in the case of baroclinical oscillations. Barotropic oscillations of a rotationally and tidally distorted gaseous sphere correspond to the radial and nonradial modes of oscillations of a spherically symmetric gaseous sphere. It is therefore possible to use Kippenhahn & Thomas approach to compute the eigenfrequencies of those barotropic modes of oscillations of the distorted model which correspond to the radial and nonradial modes of oscillations of the original undistorted model. This may be done by writing down the equations governing the radial and nonradial modes of oscillations of its topologically equivalent spherical model.

Using this approach the eigenvalue problem determining the eigenfrequencies of pseudo-radial modes of barotropic oscillations can be expressed as

$$A(n, q, r_0) \frac{d^2\eta}{dr_0^2} + \left[\frac{4-\mu}{r_0} B(n, q, r_0) - C(n, q, r_0) \right] \frac{d\eta}{dr_0} + \left[\frac{D^2 \sigma^2 \rho_\psi}{\gamma P_\psi} - \left(3 - \frac{4}{\gamma} \right) \frac{\mu}{r_0^2} E(n, q, r_0) \right] \eta = 0, \qquad \ldots(16)$$

where

$$A(n, q, r_0) = 1 - \frac{16}{3} n r_0^3 - \left(\frac{56}{5} q^2 + \frac{112}{15} nq + \frac{104}{45} n^2\right) r_0^6 - \frac{90}{7} q^2 r_0^8 - \frac{44}{3} q^2 r_0^{10} + \ldots,$$

$$B(n, q, r_0) = 1 - \frac{10}{3} n r_0^3 - \left(\frac{32}{5} q^2 + \frac{64}{15} nq + \frac{188}{45} n^2\right) r_0^6 - \frac{50}{7} q^2 r_0^8 - 8 q^2 r_0^{10} + \ldots,$$

$$C(n, q, r_0) = \frac{1}{r_0}\left[8 n r_0^3 + \left(\frac{168}{5} q^2 + \frac{112}{5} nq - \frac{104}{15} n^2\right) r_0^6 + \frac{360}{7} q^2 r_0^8 + \frac{220}{3} q^2 r_0^{10} + \ldots\right],$$

$$E(n, q, r_0) = 1 - \frac{4n}{3} r_0^3 - \left(\frac{8}{5} q^2 + \frac{16}{15} nq + \frac{92}{45} n^2\right) r_0^6 - \frac{10}{7} q^2 r_0^8 - \frac{4}{3} q^2 r_0^{10} + \ldots.$$

Also

$$\mu = -\frac{r_\psi}{P_\psi} \frac{dP_\psi}{dr_0} \frac{dr_0}{dr_\psi} = -F(n, q, r_0) \frac{r_0}{P_\psi} \frac{dP_\psi}{dr_0},$$

where

$$F(n, q, r_0) = \left[1 - 2 n r_0^3 - \left(\frac{24}{5} q^2 + \frac{16}{5} nq + \frac{72}{15} n^2\right) r_0^6 - \frac{40}{7} q^2 r_0^8 - \frac{20}{3} q^2 r_0^{10} + \ldots\right].$$

In the absence of any distortion ($n = q = 0$, $D = R$, $P_\psi = P$, $\rho_\psi = \rho$, $r_0 = x$), the above equation reduces to the usual equation determining the eigenfrequencies of small radial oscillations of a gaseous sphere. Equation (16) forms an eigenvalue problem in the frequency of oscillation, σ. This eigenvalue problem is of Sturm-Liouevìlle type having singularities both at the centre and the surface of the model. It has to be solved subject to the boundary conditions which require η to be finite at the centre as well as at the free surface. The method commonly used for obtaining eigenfrequencies of radial modes of oscillations of a spherically symmetric star can be used to solve this eigenvalue problem.

Similarly the system of differential equations which determine the eigenfrequencies of the barotropic modes which correspond to the nonradial modes of oscillations of a gaseous sphere are

$$\frac{d\zeta}{dx} + B_1\zeta + \left(B_2 + \frac{1}{\sigma^2} B_3\right)\eta + \frac{1}{\sigma^2} B_3\Phi = 0,$$

$$\frac{d\eta}{dx} + (E_1\sigma^2 + E_2)\zeta + E_3\eta + E_4\phi + \frac{d\Phi}{dx} = 0, \qquad \ldots(17)$$

and

$$\frac{d^2\Phi}{dx^2} + F_1\frac{d\phi}{dx} + F_2\zeta + F_3\eta + F_4\Phi = 0,$$

where the coefficients B_1, B_2, E_1, E_2, etc., are certain functions of the variables x, n, q, P_ψ explicitly given by

$$B_1 = \frac{l+1}{x} + \frac{1}{\gamma P_\psi}\frac{dP_\psi}{dx},$$

$$B_2 = \frac{2\pi G\rho_c}{Dx}\frac{\rho_\psi}{\gamma P_\psi} r_\psi^2 \frac{dr_\psi}{dx}$$

$$= \frac{2\pi G\rho_c}{\gamma P_\psi} D^2 \rho_\psi r_{os}^3 \times \left[1 + 4nx^3 r_{os}^3 + \left(\frac{36}{5}q^2 + \frac{72}{15}nq + \frac{864}{45}n^2\right)x^6 r_{os}^6 + \frac{55}{7}q^2x^8r_{os}^8 + \frac{26}{3}q^2x^{10}r_{os}^{10} + \ldots\right],$$

$$B_3 = -\frac{l(l+1)}{Dx}\frac{dr_\psi}{dx} 2\pi G\rho_c$$

$$= -2\pi G\rho_c\frac{l(l+1)}{x} r_{os}\left[1 + \frac{8n}{3}x^3r_{os}^3 + \left(\frac{28}{5}q^2 + \frac{56}{15}nq + \frac{532}{45}n^2\right)x^6 r_{os}^6 + \frac{45}{7}q^2x^8r_{os}^8 + \frac{22}{3}q^2x^{10}r_{os}^{10} + \ldots\right],$$

$$E_1 = -\frac{1}{2\pi G\rho_c}\frac{Dx}{r_\psi^2}\frac{dr_\psi}{dx}$$

$$= -\frac{1}{2\pi G\rho_c r_{os} x}\left[1 + \frac{4n}{3}x^3r_{os}^3 + \left(4q^2 + \frac{40}{15}nq + \frac{280}{45}n^2\right)x^6r_{os}^6 + 5q^2x^8r_{os}^8 + 6q^2x^{10}r_{os}^{10} + \ldots\right],$$

$$E_2 = \frac{1}{2\pi G\rho_c}\frac{A_\psi}{\rho_\psi}\frac{dP_{(\psi)}}{dx}\frac{Dx}{r_{(\psi)}^2}$$

$$= \frac{1}{2\pi G\rho_c D^2}\frac{1}{\rho_\psi}\left(\frac{1}{\rho_\psi}\frac{dP_\psi}{dx} - \frac{1}{\gamma P_\psi}\right)\frac{dP_\psi}{dx}\frac{1}{r_{os}^3 x}\left[1 - 4nx^3r_{os}^3 - \left(\frac{36}{5}q^2 + \frac{72}{15}nq + \frac{144}{45}n^2\right)x^6r_{os}^6 - \frac{55}{7}q^2x^8r_{os}^8 - \frac{26}{3}q^2x^{10}r_{os}^{10} + \ldots\right],$$

$$E_3 = \frac{l}{x} + A_\psi \frac{dr_\psi}{dx}$$

$$= \frac{l}{x} + \left(\frac{1}{\rho_\psi}\frac{d\rho_\psi}{dx} - \frac{1}{\gamma P_\psi}\frac{dP_\psi}{dx}\right),$$

$$E_4 = \frac{l}{x},$$

$$F_1 = \frac{2l}{x} - \frac{(d^2 r_\psi/dx^2)}{(dr_\psi/dx)} + \frac{2}{r_\psi}\frac{dr_\psi}{dx}$$

$$= \frac{2(l+1)}{x} - \frac{1}{x}[4nx^3 r_{os}^3 + (24q^2 + 16nq + 32n^2)x^6 r_{os}^6 + 40q^2 x^8 r_{os}^8$$

$$+ 60q^2 x^{10} r_{os}^{10} + \ldots],$$

$$F_2 = 2\frac{\rho_\psi}{\rho_c}\frac{A_\psi Dx}{r_\psi^2}\left(\frac{dr_\psi}{dx}\right)^2$$

$$= 2\frac{\rho_\psi}{\rho_c}\left(\frac{1}{P_\psi}\frac{dP_\psi}{dx} - \frac{1}{\gamma P_\psi}\frac{dP_\psi}{dx}\right)\frac{1}{xr_{os}}\left[1 + \frac{4n}{3}x^3 r_{os}^3 + \left(4q^2 + \frac{40}{15}nq\right.\right.$$

$$\left.\left. + \frac{280}{45}n^2\right)x^6 r_{os}^6 + 5q^2 x^8 r_{os}^8 + 6q^2 x^{10} r_{os}^{10} + \ldots\right],$$

$$F_3 = -\frac{4\pi G\rho_\psi^2}{\gamma P_\psi}\left(\frac{dr_\psi}{dx}\right)^2$$

$$= -\frac{4\pi G\rho_\psi^2 D^2 r_{os}^2}{\gamma P_\psi}\left[1 + \frac{16}{3}nx^3 r_{os}^3 + \left(\frac{56}{5}q^2 + \frac{112}{15}nq\right.\right.$$

$$\left.\left. + \frac{1384}{45}n^2\right)x^6 r_{os}^6 + \frac{90}{7}q^2 x^8 r_{os}^8 + \frac{44}{3}q^2 x^{10} r_{os}^{10} + \ldots\right],$$

$$F_4 = \frac{l(l-1)}{x^2} - \frac{l}{x}\frac{d^2 r_\psi/dx^2}{dr_\psi/dx} + \frac{2l}{xr_\psi}\frac{dr_\psi}{dx} - \frac{l(l+1)}{r_\psi^2}\left(\frac{dr_\psi}{dx}\right)^2$$

$$= -\frac{l}{x^2}\left[8nx^3 r_{os}^3 + \left(\frac{168}{5}q^2 + \frac{336}{15}nq + \frac{804}{15}n^2\right)x^6 r_{os}^6 + \frac{360}{7}q^2 x^8 r_{os}^8\right.$$

$$+ \frac{220}{3}q^2 x_{10} r_{os}^{10} + l\left\{4nx^3 r_{os}^3 + \left(\frac{48}{5}q^2 + \frac{96}{15}nq + \frac{972}{45}n^2\right)x^6 r_{os}^6\right.$$

$$\left.\left. + \frac{80}{7}q^2 x^8 r_{os}^8 + \frac{40}{3}q^2 x^{10} r_{os}^{10} + \ldots\right\}\right].$$

In the above expressions terms up to second order of smallness in n and q have been retained. Other symbols have usual meanings.

The eigenvalue problem posed by the system of differential equations (17) has to be solved subject to the boundary conditions

$$\eta + \Phi = \frac{\sigma^2}{2\pi G P_c l r_{os}} \zeta, \text{ and } \frac{d\Phi}{dx} = 0, \qquad \ldots(18)$$

at the centre $x = 0$.

The boundary conditions at the free surface ($x = 1$) are

$$2\pi G \rho_c \rho_\psi D^2 r_{os}^3 \left[1 + 4nr_{os}^3 + \left(\frac{36}{5} q^2 + \frac{72}{15} nq + \frac{864}{45} n^2\right) r_{os}^6 + \frac{55}{7} q^2 r_{os}^8 + \frac{26}{3} q^2 r_{os}^{10} + \ldots \right] \eta + \frac{dP_\psi}{dx} = 0,$$

$$\frac{d\Phi}{dx} + \left[l + \frac{l+1}{r_\psi} \frac{dr_\psi}{dx} \right] \Phi + \frac{2DP_\psi}{P_c r_\psi^2} \frac{dr_\psi}{dx} = 0,$$

and

$$\frac{d\Phi}{dx} + \left[l + (l+1) \left\{ 1 + 2nr_{os}^3 + \left(\frac{24}{5} q^2 + \frac{48}{15} nq + \frac{396}{45} n^2\right) r_{os}^6 + \frac{40}{7} q^2 r_{os}^3 + \frac{20}{3} q^2 r_{os}^{10} + \ldots \right\} \right] \Phi + \frac{2\rho_\psi}{\rho_c r_{os}} \left[1 + \frac{4n}{3} r_{os}^3 + \left(4q^2 + \frac{40}{15} nq + \frac{230}{45} n^2\right) r_{os}^6 + 5q^2 r_{os}^8 + 6q^2 r_{os}^{10} + \ldots \right] \zeta = 0. \qquad \ldots(19)$$

Thus in terms of the nondimensional eigenfunctions ζ, η and Φ, the problem determining eigenfrequencies σ of barotropic modes corresponding to the nonradial modes of the undistorted model reduces to solving the system of differential equations (17) subject to the boundary conditions (18) at the centre and the boundary conditions (19) at the free surface. Methods commonly used for computing the eigenfrequencies of nonradial modes of oscillation of spherically symmetric stars can be used to solve this eigenvalue problem.

5. Numerical illustrations

In this section we demonstrate the use of the method proposed in the earlier sections to compute the equilibrium structures and periods of certain barotropic modes of oscillations of certain primary components of binary stars. The undistorted models of the primary components of these binary systems have been assumed to have polytropic structures of index N equal to 1.5, 3, and 4. The computations have been performed for different choices of the mass ratio q and n (one half of the square of the angular velocity of rotation w). Both synchronous ($n = (q+1)/2$) as well as nonsynchronous binary systems have been considered. The results are presented in tables 1-3. Values of θ_ψ the values of polytropic index θ on the various equipotential surfaces inside the distorted models are given in table 1. Volumes and surface areas of some of these distorted models are given in table 2. The shape of the outermost equipotential surface of the primary component of a synchronously rotating system (for $N = 3.0$, $q = 0.2$, and $n = 0.6$) is given in table 2(a). Whereas the eigenfrequencies of the barotropic modes corresponding

Table 1. Values of θ_ψ for rotationally and tidally distorted polytropes as primary components of binary systems

$x = r_0/r_{0a}$	Undistorted $n = 0.0$, $q = 0.0$	Non-synchronous binary $n = 0.1$, $q = 0.2$	Synchronous binary systems $n = 0.55$, $q = 0.1$	Synchronous binary systems $n = 0.6$, $q = 0.2$
	Values of θ_ψ for $N = 1.5$			
0.0	1.00000	1.00000	1.00000	1.00000
0.1	0.97797	0.97814	0.97887	0.97895
0.2	0.91446	0.91509	0.91781	0.91811
0.4	0.69488	0.69673	0.70481	0.70570
0.5	0.56067	0.56291	0.57276	0.57384
0.6	0.42490	0.42720	0.43736	0.43849
0.8	0.18112	0.18250	0.18880	0.18949
0.9	0.08218	0.08283	0.08592	0.08623
1.0	0.00000	0.00000	0.00000	0.00000
	Values of π_ψ for $N = 3.0$			
0.0	1.00000	1.00000	1.000000	1.00000
0.1	0.92600	0.92627	0.92749	0.92758
0.2	0.75322	0.75397	0.75730	0.75755
0.4	0.40590	0.40686	0.41118	0.41148
0.5	0.28402	0.28482	0.28837	0.28860
0.6	0.19316	0.19374	0.19635	0.19649
0.8	0.07313	0.07334	0.07434	0.07434
0.9	0.03251	0.03260	0.03304	0.03300
1.0	0.00000	0.00000	0.00000	0.00000
	Values of θ_ψ for $N = 4.0$			
0.0	1.00000	1.00000	1.00000	1.00000
0.1	0.73999	0.74020	0.74123	0.74128
0.2	0.44089	0.44114	0.44234	0.44240
0.4	0.17893	0.17906	0.17966	0.17967
0.5	0.11984	0.11992	0.12030	0.12030
0.6	0.07999	0.08004	0.08027	0.08026
0.8	0.03001	0.03001	0.03006	0.03004
0.9	0.01334	0.01332	0.01333	0.01332
1.0	0.00000	0.00000	0.00000	0.00000

Table 2. Volumes and surface areas of rotationally and tidally distorted polytropes as primary components of a binary system

n	q	$N = 1.5$ Volume $\times 10^{-2}$	$N = 1.5$ Surface $\times 10^{-2}$	$N = 3.0$ Volume $\times 10^{-3}$	$N = 3.0$ Surface $\times 10^{-2}$	$N = 4.0$ Volume $\times 10^{-3}$	$N = 4.0$ Surface $\times 10^{-3}$
		Undistorted					
0.00	0.00	2.0432	1.6776	1.37474	5.9774	14.06254	2.81785
		Non-synchronous binary system					
0.10	0.20	2.0770	1.6960	1.40507	6.0664	14.43935	2.86744
0.20	0.20	2.1080	1.7128	1.43461	6.1509	14.80513	2.91559
		Synchronous binary systems					
0.55	0.10	2.2214	1.7744	1.54860	6.4754	16.25581	3.10452
0.60	0.20	2.2437	1.7862	1.57089	6.5375	16.53994	3.14066

Table 2a. Shape of the outermost equipotential surface of the primary component of a synchronously rotating binary system for ($N = 3.0$, $q = 0.2$ and $n = 0.6$)

Section by plane through the axis of rotation and the line joining the mass centres of the companions ($\phi = 0$)				Section by plane through the axis of rotation and perpendicular to the line joining the mass centres of the primary ($\phi = 90°$)			
$\theta°$	r/D	$\theta°$	r/D	$\theta°$	r/D	$\theta°$	r/D
0	0.48927	0	0.48927	0	0 48927	0	0 48927
15	0 49078	−15	0 49392	15	0 49157	−15	0 49157
30	0.49874	−30	0 50403	30	0 49821	−30	0.49821
45	0.50719	−45	0.51071	45	0.50813	−45	0 50813
60	0.53796	−60	0.53320	60	0.51903	−60	0.51903
75	0.56269	−75	0 54552	75	0.52762	−75	0.52762
90	0.57366	−90	0.55032	90	0.53089	−90	0.53089

(Shape is symmetrical about plane through the centres of mass and perpendicular to axis of rotation).

Table 3a. Eigenfrequencies $\omega^2 (= r_\alpha^3 D^3 \sigma^2 / GM_0)$ for the fundamental mode (ω_0^2) and the first mode (ω_1^2) of pseudo-radial oscillations for ($\gamma = 5/3$)

n	q	k	$N = 1.5$		$N = 3.0$		$N = 4.0$	
			ω_0^2	ω_1^2	ω_0^2	ω_1^2	ω_0^2	ω_1^2
			Undistorted model					
0.00	0.00	1.0	2.07060	12.5325	9.2547	16.9831	15 1490	24 9452
			Non-synchronous binary systems					
0.10	0.20	0.5	2.6486	12.1236	9.0700	16.4804	14.7204	24.1236
0.20	0.20	0.5	2.5938	11.7414	8.8881	16.0026	14.2981	23.3390
			Synchronous binary systems					
0.550	0.100	0.5	2.4024	10.3896	8.2133	14.2748	12.7428	20.4361
0.600	0.200	0.5	2.3704	10.1218	8.0942	13.9438	12.4463	19.8369

to the radial modes of oscillation are presented in table 3(a), the eigenfrequencies corresponding to the nonradial modes of oscillations are presented in table 3(b).

6. Concluding remarks

The techniques proposed in this paper can be used to compute the equilibrium structure and periods of certain barotropic modes of oscillations of stars which are the primary components of binary systems. These techniques can be easily incorporated into the computer software commonly available for computing the equilibrium structure and periods of radial and nonradial modes of oscillations of the spherically symmetric stars.

The techniques however have certain limitations. These can only be used in the case of the primary component of a binary system. It is also assumed that the mass of the secondary and the angular velocity of rotation are not unduly large so that terms beyond second order of smallness in q and n can be neglected. It is also assumed that the deviation of the shape of the primary star under investigation from spherical symmetry is not too large and that it is reasonably centrally condensed so that its equipotential

Table 3b. Eigenvalues ω^2 ($= r_\alpha^3 D^3 \sigma^2 / GM_0$) for the nonradial modes of polytropic models of indices 1 5, 3 and 4 for $l = 2$ and $\gamma = 5/3$

Mode	Undistorted	Non-synchronous binary systems		Synchronous binary system
	$n = 0.0$, $q = 0.0$, $k = 1.0$	$n = 0$ 1, $q = 0.1$, $k = 0$ 5	$n = 0$ 2, $q = 0$ 2, $k = 0.5$	$n = 0$ 55, $q = 0.1$, $k = 0.5$
		Polytrope of index 1.5		
p_3	41.2978 (3-3)	39 9971	38.5917	34.2883
p_2	23.5164 (2-2)	22 7365	21.8989	19.3390
p_1	10.2869 (1-1)	9.9421	9.5729	8.4433
f	2.1204 (0-0)	2.0542	1 9841	1 7644
		Polytrope of index 3		
p_3	41.4721 (3-3)	40.0323	38.4805	33 5956
p_2	26 7220 (2-2)	25.8206	24 8508	21.7848
p_1	15.2629 (1-2)	14.7918	14.2883	12.8817
f	8.1749 (0-0)	8 0304	7.8797	7 4648
g_1	4 9152 (1-1)	4.8733	4.8314	4 7324
g_2	2.8320 (2-2)	2.8125	2 7923	2 7230
g_3	1.8335 (3-3)	1.8202	1.8087	1.7641
		Polytrope of index 4		
p_3	62.8802 (5-5)	61.0969	59.1526	53.1433
p_2	50.8106 (4-4)	49 7159 (4-6)	48.5281 (4-6)	44.6485 (4-6)
p_1	42.1344 (3-5)	41.2448 (5-5)	40.2439 (5-5)	36.9333 (5-5)
f	34.3261 (4-4)	33.6892	32.9768	30.4995 (4-6)
g_1	27.6387 (3-5)	27.1346	26.6083 (5-5)	24.9989 (5-5)
g_2	23 0039 (4-4)	22 7000	22.3212 (4-8)	20.8000 (4-6)
g_3	18.3057 (5-5)	18.1311	17 9549	17.4096

(Numbers shown in parenthesis along the eigenfrequencies are the number of nodes appearing in the eigenfunctions ζ and η. In the case of the entries in a row where no such nodes are shown indicates that these eigenfrequencies also have the same number of nodes in ζ and η as are shown in the second column of this table for the undistorted case).

surfaces can be approximated by appropriate Roche equipotential surfaces. These approximations are in fact valid in the case of a majority of the observed binary systems.

References

Fitch, W. S. (1969) *Ap. J.* **158**, 269.
Kopal, Z. (1972) *Adv. Astr. Ap.* **9**, 1.
Kopal, Z. (1974) *Ap. Sp. Sci.* **27**, 389.
Kippenhahn, R. & Thomas, H. C. (1970) in *Stellar rotation* (ed. : A. Slettbak) Reidel, p. 20.
Mohan, C. & Saxena, R. M. (1983, 1985) *Ap. Sp. Sci.* **95**, 369; **113**, 155.

Mohan, C & Singh, V. P. (1982) *Ap. Sp Sci* **85**, 83.
Mohan, C & Agarwal, S R (1987) *Ap Sp. Sci* **129**, 73

Discussion

Kaul : Have these models been applied to close binary systems?

Mohan : The method can be applied to close binary systems as long as the basic assumptions of the Roche model are not violated.

Kaul : Have they been compared with other models?

Mohan : Yes, we have compared our results with those obtained by other authors in the case of rotating stars (for which the results are normally available) by taking $q = 0$. The results by our method compare well with the results obtained by other methods used for incorporating effects of rotation. This has been shown in one of our earlier publications which has appeared in the *Ap. Sp. Sci.* 1985 issue. We are not aware of any results in which combined effects of rotation and tidal distortions have been included in theoretical investigations.

Bhat : Have the results of your calculations been applied to a physical system? How would the results be affected for a contact binary system?

Mohan : The results have been applied to 16 Lac, which is the primary component of a binary system and is also a pulsating variable. Fitch has reported three of its observed frequencies. We tried to see if any of the three polytropic models considered by us, namely $n = 1.5$, 3 and 4, could give pulsation periods which are close to the observed periods. It was found that even though none of the three polytropic models give all the three observed periods, but a better fit could perhaps have been achieved for a polytropic index between three and four.

As regards the second part of the question the method should be used with caution, since we assume that the primary is within its Roche Lobe and the secondary is a point mass not on the surface of the primary but some distance away from it.