Bull. Astr. Soc. India (1986) **14**, 1–7

General upper limits to the age of the universe*

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Received 1985 December 9

Abstract. We model here inhomogeneous and anisotropic cosmologies by means of a general class of relativistic spacetimes; and the past extension of timelike galaxy worldlines from the present epoch is examined under reasonable assumptions. Conclusions are derived concerning the most general upper bounds that can be set on the age of the universe within this framework.

Key words: cosmology—globally hyperbolic spacetime—age of universe

1. Introduction

The standard cosmological picture describes the observable large scale universe in terms of the Friedmann-Robertson-Walker models which are based on the assumption of the cosmological principle stating that the universe is homogeneous and isotropic. This means that the universe possesses an unlimited degree of uniformity in matter distribution in all directions and regions of space. These constraints uniquely fix the geometry of the universe and the picture here is that of three-spacelike hypersurfaces of constant spatial curvature evolving in time. This scenario then predicts that all the particle trajectories and world lines must originate from a big-bang singularity at finite time in the past. The total time elapsed from this initial singularity to the present epoch which is represented by a spacelike hypersurface S_0 then gives the age of the universe:

$$t_{\rm age} = H_0^{-1} f(q_0). \qquad ...(1)$$

The quantities H_0 and q_0 here are the parameters characterizing Friedmann cosmologies (see *e.g.* Weinberg 1972). $f(q_0)$ is a monotonic decreasing function of q_0 which attains its maximum value of unity as q_0 approaches zero and tends to zero as q_0 becomes infinitely large. Since the observational data are extremely uncertain

*Received 'honourable mention' at the 1985 Gravity Research Foundation essay competition.

regarding the value of q_0, the only definite statement that could be made about $t_{\rm age}$ is that H_0^{-1} provides a sheer upper limit to the age of the universe. There is no certainty again regarding the value of H_0, however the present consensus seems to be for H_0 between 40–120 km s^{-1} Mpc^{-1} (Narlikar 1983). This corresponds to the upper bounds on age given by

$$t_{\max} = \begin{cases} 24.4 \times 10^9 \text{ yr for } H_0 = 40 \text{ km s}^{-1} \text{ Mpc}^{-1} \\ 8.1 \times 10^9 \text{ yr for } H_0 = 120 \text{ km s}^{-1} \text{ Mpc}^{-1}. \end{cases} \qquad \ldots(2)$$

However, it is well recognized that the assumption of exact homogeneity and isotropy are far-reaching idealizations of the real picture. Actually, we observe the clustering of matter on all scales and real inhomogeneities may be present at all stages in the universe (Dicke & Peebles 1979). Friedmann models cannot be taken as an exact situation but it is an approximation that may hold on a large scale. But the currently available data in observational cosmology are of uncertain accuracy and in many cases there are many unknown factors involved so that it is very difficult to answer this as well as many other questions in a definite way. As for the early universe, whereas the homogeneity may be obtained in a 'patchwise' manner, the isotropy is open to question (Barrow & Turner 1981), and in fact some chaotic initial state would seem more natural. Actually, the isotropic models form only a 'set of measure zero' amongst permitted relativistic cosmologies and great attention has been paid to analyse inhomogeneous and anisotropic cosmologies (MacCallum 1979). Assuming uniformity over the hypersurface, to which we do not have much observational access, is a big extrapolation. This postulated uniformity is surely lacking on small scales, and many entertain doubts as to its validity even on a smoothed-out large scale.

Thus, it would be very useful to study the important questions such as the age of the universe without demanding the exact conditions of homogeneity and isotropy and in as much model independent way as is possible. We consider here a class of general relativistic spacetimes without recourse to the assumptions of homogeneity and isotropy and derive conclusions concerning the most general upper bounds that can be set on the age of the universe. Thus, the universe need not be of exact Friedmann type, and a knowledge of the cosmological parameters H_0 and q_0 is not required which are any way uncertain at the moment.

2. Timelike trajectories of matter in a general spacetime

We model here an inhomogeneous and anisotropic universe by mean of a general globally hyperbolic spacetime. These are spacetimes which admit a spacelike hypersurface S, called a Cauchy surface, the data on which can be evolved into the future (past) to predict the future (past) states of the universe. Geroch (1970) showed that such a spacetime can be covered by a one-parameter family (S_t) of spacelike Cauchy surfaces. Thus the state of the universe at any given epoch t can be referred to in terms of surface of cosmic simultaneity t = constant. We note that the well-known cosmological spacetimes such as Friedmann-Robertson-Walker models, Bianchi, or steady state cosmologies are globally hyperbolic.

We would like to investigate within this globally hyperbolic framework the extension into the past of timelike geodesic trajectories by considering the gravitational focusing effect on the matter in a spacetime. Let the present epoch be characterized by a spacelike global Cauchy surface S_0, where we set $t = 0$, and the matter distribution on S_0 be given by a second rank stress-energy tensor T_{ij} which satisfies the usual energy condition that there are no negative energy fields in the spacetime

$$(T_{ij} - \tfrac{1}{2} g_{ij}T) V^i V^j \geqslant 0, \qquad \ldots(3)$$

where V^i is a unit timelike vector. Further, let the dynamics of the universe be governed by the Einstein's equations

$$R_{ij} = 8\pi G(T_{ij} - \tfrac{1}{2}g_{ij}T). \qquad \ldots(4)$$

The last assumption means, using equation (3), that we have

$$R_{ij}V^iV^j \geqslant 0 \qquad \ldots(5)$$

for all matter fields. Though we have not required the homogeneity or isotropy of material distribution on S_0, we shall be assuming that there exists a minimum for density distribution on S_0, which, in view of the observed expansion of the universe, should exhibit a nondecreasing behaviour in the past. This means that there exists some $k > 0$ such that

$$R_{ij}V^iV^j \geqslant k > 0 \qquad \ldots(6)$$

at the present and all past epochs.

The gravitational focusing effect on matter in a spacetime can be characterized by the concept of a point conjugate to a spacelike hypersurface S_t along a timelike geodesic $\gamma(t)$, orthogonal to S_t. Consider a congruence of timelike geodesics orthogonal to S_t. Let $\gamma(t)$ be a member of the congruence, then a point q along $\gamma(t)$ is said to be conjugate to S_t if neighbouring timelike geodesics orthogonal to S_t intersect at q. Such a situation arises when the expansion θ of the congruence becomes infinite at q, which is governed by the Raychaudhuri equation (Hawking & Ellis 1973)

$$\frac{d\theta}{dt} = - R_{ij}V^iV^j - 2\sigma^2 - \frac{1}{n}\,\theta^2, \qquad \ldots(7)$$

where σ is the shear of the congruence, which is intrinsically positive. Here $n = 3$ for timelike geodesics and $n = 2$ for null geodesics. A timelike geodesic $\gamma(t)$ will be orthogonal to S_0 provided the expansion θ along $\gamma(t)$ satisfies $\theta = \chi^a_a$ at S_0, where χ_{ab} is the second fundamental form of the spacelike surface S_0. It is easy to see that if we substitute $\theta = z^{-1}(dz/dt)$ with $z = x^2$, then equation (7) becomes (Tipler 1976)

$$\frac{d^2x}{dt^2} + F(t)x = 0, \qquad \ldots(8)$$

where $F(t) = \frac{1}{3}(R_{ij}V^iV^j + 2\sigma^2)$. Then the problem of finding a point q conjugate to S_0 along $\gamma(t)$ becomes that of finding a solution $x(t)$ to equation (8) which vanishes at q. Specifically, for the orthogonal timelike geodesic $\gamma(t)$, a point q along $\gamma(t)$ will be conjugate to S_0 provided a solution $x(t)$ of equation (8) satisfying the initial conditions

$$x(0) = \alpha, \quad \left(\frac{dx}{dt}\right)_0 = \alpha\chi^a_a \qquad \ldots(9)$$

vanishes at q. In order to analyse the occurrence of zeros in the solutions to equation (8) we shall use the Sturm comparison theorem on the solution of second order differential equations (*see e.g.* Hille 1969) which compares the distribution of zeros of the solutions $u(t)$ and $v(t)$ of the equations

$$\left.\begin{aligned} \frac{d^2u}{dt^2} + G_1(t)u &= 0, \\ \frac{d^2v}{dt^2} + G_2(t)v &= 0, \end{aligned}\right\} \qquad \ldots(10)$$

where $G_1 \leqslant G_2$ in an interval (a, b). The theorem then shows that if $u(t)$ has m zeroes in $a < t \leqslant b$, then $v(t)$ has at least m zeroes in the same interval and the ith zero of $v(t)$ is less than the ith zero of $u(t)$.

Now let $k^2 = \min F(t) = \min \frac{1}{3}(R_{ij}V^iV^j + 2\sigma^2)$, and consider

$$\frac{d^2x}{dt^2} + k^2x = 0. \qquad \ldots(11)$$

Then applying the Sturm theorem to equations (8) and (11) we see that if the solution to equation (11) satisfying the initial conditions (9) has a zero in the interval $0 < t \leqslant t_1$, then the solution of equation (8) defined by the same initial conditions must have a zero in the same interval which must occur before the zero of the solution of eauation (11). The general solution of equation (11) can be written as

$$x = A \sin (B + kt). \qquad \ldots(12)$$

Let us choose the initial conditions as

$$x(0) = \frac{1}{(\chi^{a2}_a + k^2)^{\frac{1}{2}}}; \quad \left(\frac{dx}{dt}\right)_0 = \frac{\chi^a_a}{(\chi^{a2}_a + k^2)^{\frac{1}{2}}}, \qquad \ldots(13)$$

where χ^a_a is negative valued on S_0 since the universe is expanding everywhere. It may be possible to envisage scenarios in which the universe might be expanding at some places on S_0 and contracting in some other regions of S_0; however we shall not consider such possibilities here. It is then easy to see that the corresponding solution of equation (11) is given as

$$x = \frac{1}{k} \sin (\theta - kt), \qquad \ldots(14)$$

with

$$\theta = \sin^{-1}\left\{\frac{k}{(\chi_a^{a2} + k^2)^{1/2}}\right\}. \qquad \ldots(15)$$

Thus we have $0 < \theta \leqslant \pi/2$ and a zero for x must occur within $0 < kt \leqslant \pi/2$. Then, using the comparison theorem, we see that the solution of equation (8) as defined by the initial conditions (13) must vanish within an interval of time $0 < t \leqslant \pi/2k$, *i.e.* if $\gamma(t)$ is any timelike geodesic orthogonal to S_0, then there must be a point q on $\gamma(t)$, conjugate to S_0, within the above interval.

3. Maximum age of the universe

It is now possible to investigate in general the past extensions of arbitrary timelike trajectories Γ from the present epoch S_0 (Joshi & Chitre 1981). Let p be an event on S_0 and Γ be a past directed, endless timelike curve from $p = \Gamma(0)$. Suppose Γ can be extended to arbitrary parameter values in the past, then choose $q = \Gamma(\pi/2k)$ to be an event on Γ. Then by a well-known property of globally hyperbolic spacetimes (Hawking & Ellis 1973), there exists a timelike geodesic γ from q orthogonal to S_0 along which the proper time-lengths of all nonspacelike curves from q to S_0 are maximized and further, γ does not contain any conjugate point between q and S_0. However, as shown above, any timelike geodesic $\gamma(t)$ orthogonal to S_0 which is as long as $\pi/2k$ in the past must contain a point conjugate to S_0 within that interval, which is not possible. Consequently, we conclude that no timelike curve from S_0 can be extended into the past beyond the proper time length $\pi/2k$.

The above results can be employed to obtain general upper bounds to the age of a globally hyperbolic universe. Taking the stress-energy tensor of the form

$$T_{ij} = (\rho + p)\, u_i u_j + p g_{ij}, \qquad \ldots(16)$$

and using Einstein's equations (6), we get

$$\begin{aligned} R_{ij}V^iV^j &= 8\pi G(T_{ij}V^iV^j + \tfrac{1}{2}T) \\ &= 8\pi G(\rho + p)\,(V^4)^2 - 4\pi G\rho + 4\pi G p, \end{aligned}$$

i.e.
$$R_{ij}V^iV^j \geqslant 4\pi G(\rho + 3p). \qquad \ldots(17)$$

If the pressure p is neglected, then we get

$$k^2 = \min \tfrac{1}{3}(R_{ij}V^iV^j + 2\sigma^2) \geqslant \frac{4\pi G\rho}{3}; \qquad \ldots(18)$$

and the maximal possible extension for any timelike worldline from the present epoch into the past, or the maximal possible age of the universe, is given by

$$t_{\max} = \frac{\pi}{2}\left(\frac{3}{4\pi G\rho}\right)^{1/2} = \pi\left(\frac{3}{16\pi G\rho}\right)^{1/2} \qquad \ldots(19)$$

within the framework of general globally hyperbolic spacetimes. In the case of radiation-dominated models we can take $p = \rho/3$ and we have

$$t_{max} = \frac{\pi}{2}\left(\frac{3}{8\pi G\rho}\right)^{1/2} = \pi\left(\frac{3}{32\pi G\rho}\right)^{1/2}. \qquad ...(20)$$

The relationships (19) and (20) provide upper limits to the age even when allowing for departures from homogeneity and isotropy. It should be noted that the ρ occurring in equation (19) is a global minimum taken over the epoch of constant time for the averaged out density occurring in equation (16). The average mass density as indicated by the visible galaxies is about 10^{-30} gm cm^{-3}. However, as such, the entire subject of mass-energy density of universe is under active investigation and it is believed that about ten times the visible mass may be in some invisible form in the intergalactic or intragalactic mediums (Faber & Callagher 1979). The x-ray observations strongly favour the existence of a hot ionized intergalactic gas within the cluster of galaxies whereas weakly interacting massive neutrinos could be another source. We may have to wait for further observations which might determine the contributions from these sources. However, if a hierarchical clustering of matter stops at any stage, such as, say, clusters of galaxies, then the above density minimum would be achieved. In any case, if one accepts the microwave background radiation (MBR) as having some kind of global origin, then ρ_{MBR} provides a firm lower limit to the ρ_{min} sought for and the firm general upper limit to the age of the universe as given by equation (20) is

$$t_{max} = \pi\left(\frac{3}{32\pi G\rho_{MBR}}\right)^{1/2} = 3.2 \times 10^{12}\ \text{yr}, \qquad ...(21)$$

where $\rho_{MBR} = 4.4 \times 10^{-34}$ gm cm^{-3}.

Next, if we want to take the contribution by matter into account, we have to choose an entire range of densities as suggested by the above mentioned possibilities. The average matter density arising from all possible sources is believed to be anywhere between 10^{-30} to 10^{-28} gm cm^{-3} (Peebles 1979). The general upper limits given by equation (19) are given in table 1. A comparison of these results with the Friedmann bounds as given by equation (2) shows that inspite of their generality the global upper limits are quite tight and interesting. This suggests several further applications of the formalism developed here, such as deducing limits on the maximum amount of dark matter density that could be accommodated in the

Table 1. Maximum possible age of the universe as a function of mass-energy density

Matter (10^{-30} gm cm^{-3})	t_{max} (10^{10} yr)
1	9.43
4	4.72
8	3.34
12	2.72
16	2.36
20	2.11
30	1.72
40	1.49
60	1.22
80	1.05
100	0.94

universe or on particle masses such as neutrinos or axions in a model independent way. We also note another implication obtained here that is, nonvanishing anisotropy in the universe will contribute towards reducing the age. These topics will be taken up in a later work.

4. Concluding remarks

In conclusion we would like to note the profound implications that the global methods have in analysing the structure of spacetime towards obtaining general and very useful results. An example of the above is the well-known singularity theorems in general relativity. We have shown here that it is possible to obtain significant exact results also on such important problem as the age of the universe from general global considerations.

Acknowledgements

It is a pleasure to thank Professors J. V. Narlikar and S. M. Chitre for comments.

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