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Asymmetry in energy transfer in a collision of galaxies

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Abstract. The time dependence of binding and translational energies of a pair of galaxies during head-on collisions has been studied with impulsive approximation. The two galaxies are assumed to be identical with mass distribution represented by that of a polytrope of index $n = 4$. The results indicate that more than 80 per cent of the increment in the binding energy takes place when the centres of the two galaxies are separated by a distance less than three times the median radius of a galaxy. The change in the binding energy is asymmetric with respect to the distance of closest approach, the increment in the second half of the collisions being about double that of the first half.

Key words : galaxies — collisions — tidal force — stellar dynamics

1. Introduction

In a collision of two galaxies, energy is transferred from translational motion of the galaxies to the internal motions of the constituent stars. This effect has been studied by several workers. Recently Toomre (1977) and Ahmed (1979) have made estimates for this effect for a head-on collision of galaxies using impulsive approximation and have derived simple analytical expressions for the energy transfer for galaxies represented by Plummer's models. These studies have been directed towards obtaining the energy transfer for a complete collision and do not give information about how the energy varies with time.

It is obvious that the energy change is not uniform. The tidal force is of impulsive nature, and therefore most of the energy transfer from orbital motion of the galaxies to the internal motions of the stars takes place when the two galaxies are at their closest approach. It is for this reason that many workers found it convenient to

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estimate the energy transfer by assuming uniform translational velocity corresponding to that at the closest approach throughout the encounter. A more accurate treatment should take into account the variation in the velocity of the two galaxies arising not only because of the varying gravitational attraction of the two galaxies but also because of the force due to the dynamical friction, the latter arising from the inelastic nature of the collision.

Another important aspect of the problem to be noted in this connection is that the energy transfer is not symmetrical with respect to the closest approach. Alladin *et al.* (1975) and Sastry & Alladin (1977) made the simplifying assumption that the energy transfer is symmetrical about the distance of closest approach. This assumption is rather a crude one for close collisions because in such cases the energy transfer in the second half of the encounter is much more than in the first half.

It is the aim of the present paper to study in detail the time dependence of binding energy and translational energy of galaxies during a head-on encounter of a pair of galaxies.

2. Model of galaxies

Following Alladin (1965) and Rood (1965), we shall treat galaxies as spherically symmetric distributions of stars having a density distribution of a spherical polytrope of index $n = 4$. We neglect interstellar gas. This is also a good assumption for elliptical galaxies since observations show that most of the normal ellipticals contain no detectable neutral hydrogen (Knapp *et al.* 1978). Some workers (*e.g.*, White 1978, Roos & Norman 1979) assume isothermal models of galaxies which is a good model for galaxies with extended halos. Obviously the tidal effects obtained with this model will be greater than those predicted from the polytropic model.

The density $\rho(r)$ and potential $\mathcal{V}(r)$ due to a polytropic sphere of radius R and index n at any point r are given by (Limber 1961) :

$$\rho_n(s) = \rho_{c,n} \; \Theta_n^n(s) \qquad \dots (1)$$

$$\mathcal{V}_n(s) = -\frac{GM}{R} \; \Phi(n, s). \qquad \dots (2)$$

Here

$$\Phi(n,s) = 1 + 3\,\frac{(\rho_{c,n}/\bar{\rho}_n)}{\xi_{1,n}^2}\;\Theta_n^n(s), \quad s < 1$$

$$= \frac{1}{s}, \quad s \geqslant 1 \qquad \dots (3)$$

$$s = r/R$$

and

$$\Theta_n(s) = \theta_n\;(\xi = s\xi_{1,n});$$

θ_n is the usual Lane-Emden function, $\bar{\rho}_n$ the mean density, $\rho_{c,n}$ the central density and $\xi_{1,n}$ the value of ξ (polytropic radial variable) at which $\theta_n(\xi)$ has its first zero.

The mass distribution is given by

$$\frac{dM}{ds} = 4\pi\, \rho_{c,n}\, s^2\, \Theta_n^n\,(s) \qquad \dots (5)$$

$$\frac{M(s)}{M} = \frac{\left[\xi^2\left(\frac{d\theta_n}{d\xi}\right)\right]_{\xi = s\xi_{1,n}}}{\xi_{1,n}^2\left(\frac{d\theta_n}{d\xi}\right)_{\xi_{1,n}}}. \qquad \dots (6)$$

Limber (1961) has tabulated $\Phi(n, s)$ and $\Theta_n(s)$ for different polytropes. In a spherical polytrope of index $n = 4$ and radius R, 99.9 per cent of the mass lies within $0.6R$ and 50 per cent within $0.135R$. We shall designate the radius of the polytrope by R and let $R_0 = 0.60R$. We shall consider tidal effects on stars up to R_0. If a galaxy is of $10^{11}M_\odot$ and if $R = 10$ kpc, the root mean square velocity of the stars is, by virial theorem, equal to 360 km s^{-1}. The circular and escape velocities at various distances from the centre for this case are tabulated by Sastry (1972). The forces $f(r)$ in his Table 2 should be multiplied by $\sqrt{2}$. It may be noted that $V_{\text{r.m.s.}} = V_{\text{esc}}$ at $R = R_0$.

We also need the potential energy between the two galaxies due to this mutual gravitational interaction. The potential energy between two spherical configurations of masses M_1 and M_2, each of polytropic index $n = 4$, may be written as

$$W(s) = -\frac{GM_1M_2}{R_2}\,\chi\,(\epsilon,\ s,\ n_1,\ n_2) \qquad \dots (7)$$

where $\epsilon = R_1/R_2$. The function $\chi\,(\epsilon, s, n_1, n_2)$ is tabulated by Potdar & Ballabh (1974) for polytropes of index $n = 4$ for interpenetrating cases. If the galaxies do not interpenetrate, then

$$\chi = 1/s = R_2/r. \qquad \dots (8)$$

3. Assumption of impulsive approximation

We shall make estimate for the energy transfer from the translational motion of the galaxies to the internal motions of the stars assuming that the motions of stars can be neglected as the galaxies pass through each other. This is the impulsive approximation first used by Spitzer (1958) to estimate the tidal effects of an intersteller cloud on a star cluster. The use of this approximation gives good estimate for the change in energy of a star if the time of collision of the galaxies is much larger than the orbital period of the star. Its use in the case of a slow collision in which the collision time of the galaxies is of the same order of magnitude as the orbital time of a star will in general give much error in the estimate for the energy imparted to a 'single' star. Nevertheless its use even in the case of slow but non-merging collisions is justified by the fact that in so far as the total energy change of 'all' the stars is concerned, the approximation gives reasonable values, especially if the stellar orbits are circular and have circularly symmetric distribution of velocities (Sastry & Alladin 1970).

As emphasized by White (1979) the tidal coupling in collisions is strongly dependent on the rotational properties of the galaxies. It is enhanced if the stars revolve in the direction of the revolution of the galaxies and is suppressed if they revolve in the retrograde direction. If in a galaxy there are stars moving both in direct as well as retrograde directions in equal numbers, the impulsive approximation underestimates the tidal effects in the first case and overestimates in the second case with the result that the hypothetical star which does not move at all experiences an intermediate tidal effect. Therefore the impulsive approximation serves as a useful first approximation.

In the present work we shall make estimates for the tidal effects in head-on collisions in which the collision velocity of the galaxies is not so small as to make them merge quickly nor so large that the tidal effects are unimportant. Toomre (1977) notes that the minimum velocity necessary for non-merging of two identical galaxies in a head-on collision deduced from the impulsive approximation is in good agreement with that obtained from detailed N-body calculations performed by van Albeda & van Gorkom. It is therefore expected that the results obtained here would be useful as a first approximation with which the results of detailed calculations may be compared. We would particularly like to emphasize that the asymmetric nature of energy transfer which is an interesting feature of penetrating collisions of galaxies can be deduced even from the impulsive approximation.

4. Basic equations

We shall determine the changes in the velocities of stars in the galaxy of mass M_2 (test galaxy) during a collision. Let the perturbing galaxy be of mass M_1. We define

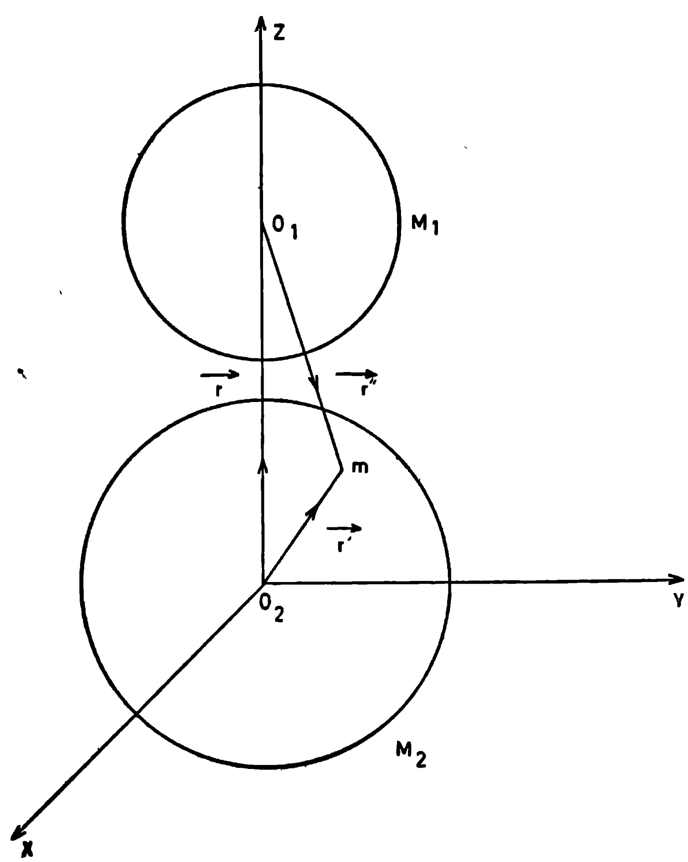

Figure 1. Positions of the two galaxies and the star.

a Cartesian co-ordinate system centred at the centre of mass of M_2 and we consider the motion of M_1 with respect to M_2 (Figure 1). Let $\mathbf{r}$ (x, y, z) denote the position of M_1 and $\mathbf{r}'$ (x', y', z') of a representative star in M_2. Let $\mathbf{r}'' = \mathbf{r}' - \mathbf{r}$.

We reckon time from the instant of initial separation. The change in velocity $\Delta\mathbf{V}'$ at any time for a representative star is given by

$$\Delta\mathbf{V}'(t) = \int_0^t \mathbf{f}_T \, dt \qquad \text{...(9)}$$

where $\mathbf{f}_T$ is the tidal force per unit mass on the star. $\mathbf{f}_T$ is obtained from polytrope theory as discussed in Sastry & Alladin (1970). For galaxies of unequal dimensions, equation (9) may be written in scalar form as :

$$\Delta V'_x(t) = -\frac{GM_1}{R_2^2} \int_0^t \left[\frac{(x - x')}{\epsilon^2 r''} \frac{d\Phi_1}{ds''} - \frac{x}{r} \frac{d\chi}{ds} \right] dt \qquad \text{... (10)}$$

$$\Delta V'_y(t) = -\frac{GM_1}{R_2^2} \int_0^t \left[\frac{(y - y')}{\epsilon^2 r''} \frac{d\Phi_1}{ds''} - \frac{y}{r} \frac{d\chi}{ds} \right] dt \qquad \text{... (11)}$$

and

$$\Delta V'_z(t) = -\frac{GM_1}{R_2^2} \int_0^t \left[\frac{(z - z')}{\epsilon^2 r''} \frac{d\Phi_1}{ds''} - \frac{z}{r} \frac{d\chi}{ds} \right] dt \qquad \text{... (12)}$$

where $s = r/R_2$, $s = r/R_1$. For a head-on collision of galaxies along the z-axis, $x = 0$, $y = 0$, so that equations (10) – (12) reduce to

$$\Delta V'_x(t) = \frac{GM_1}{\epsilon^2 R_2^2} \int_0^t \frac{x'}{r''} \frac{d\Phi_1}{ds''} \, dt \qquad \text{... (13)}$$

$$\Delta V'_y(t) = \frac{GM_1}{\epsilon^2 R_2^2} \int_0^t \frac{y'}{r''} \frac{d\Phi_1}{ds''} \, dt \qquad \text{... (14)}$$

and

$$\Delta'_z(t) = -\frac{GM_1}{R_2^2} \int_0^t \left[\frac{(z - z')}{\epsilon^2 r''} \frac{d\Phi_1}{ds''} - \frac{z}{r} \frac{d\chi}{ds} \right] dt. \qquad \text{... (15)}$$

The quantity r is obtained as a function of time for rectlinear orbit of two galaxies from equation (20) which takes into account the inelastic nature of the collision.

If $\mathbf{V}'_i$ and $\mathbf{V}'_f$ are the initial and final velocities of a star, then the change in the kinetic energy of the star per unit mass during the encounter is given by

$$\tfrac{1}{2} (\mathbf{V}'^2_f - \mathbf{V}'^2_i) = \mathbf{V}'_i . \, \Delta\mathbf{V}' + \tfrac{1}{2} (\Delta\mathbf{V}')^2. \qquad \text{... (16)}$$

The right hand side of this equation consists of a fluctuating term, $\mathbf{V}_i'.\Delta\mathbf{V}'$ and a secular term $(\Delta\mathbf{V}')^2$. For most stars $\frac{\Delta\mathbf{V}'}{\mathbf{V}'}$ is small and therefore $\mathbf{V}_i'.\Delta\mathbf{V}'$ is much larger in absolute magnitude than the $(\Delta\mathbf{V}')^2$ term. But if the stars have random motions and if they do not escape, the average of the fluctuating term is zero. Richstone (1975) studied collisions of galaxies using King's model which contains many elongated orbits of stars. He found that stars escape preferentially when $(\mathbf{V}'.\Delta\mathbf{V}') > 0$. Since $\langle \mathbf{V}'.\Delta\mathbf{V}' \rangle = 0$ for all stars, those remaining in the galaxies have $(\mathbf{V}'.\Delta\mathbf{V}') < 0$ and $|\mathbf{V}'.\Delta\mathbf{V}'| \gtrsim (\Delta\mathbf{V}')^2$. Thus in the cases studied by him the energy change is dominated by the effects of the first term. Knobloch (1978) studied the effects of weak tidal interactions between galaxies using the Fokker-Planck equation. His model of galaxies also consisted of elongated orbits of stars. He found that the stars are preferentially stripped from the outer regions and the effect of the fluctuating term in the tidal force is qualitatively similar to that of the secular trem. Dekel *et al.* (1980) noted from their N-body simulations of slow encounters of galaxies that the effects of the fluctuating term are more important if many stars are in elongated orbits while the effects of the secular term are more important if many stars are in circular orbits.

In head-on collisions of galaxies the impulsive approximation gives little or no mass loss (Sastry & Alladin 1977). Estimate of mass less from a study of the orbits of stars assumed initially circular shows that less than 2 per cent of the mass escapes even in the case of a slow head-on collision in which the initial relative velocity of the galaxies is nearly equal to the capture velocity (Ahmed 1979). We shall therefore assume $\langle \mathbf{V}.\Delta\mathbf{V} \rangle = 0$ as was done by Spitzer (1958) and Toomre (1977).

We divide the test galaxy into shells of stars, each shell being characterized by a common distance r from the centre of M_2. If each shell has N sample stars, the average change in the kinetic energy of each shell is obtained from

$$\Delta T(s') = \frac{1}{2}\langle(\Delta\mathbf{V}'(s'))^2\rangle$$

$$= \frac{1}{2N}\sum_{j=i}^{N}[(\Delta V_x')_j^2 + (\Delta V_y')_j^2 + (\Delta V_z')_j^2]. \qquad \dots (17)$$

The use of impulsive approximation implies that

$$\Delta T = \Delta U \qquad \dots (18)$$

where U is the binding energy of the galaxy. The change in the binding energy of the galaxy is obtained by integrating equation (17) over mass :

$$\Delta U = \int_0^t \langle\Delta T(s')\rangle \frac{dM_2}{ds'}\, ds' \qquad \dots (19)$$

where dM_2/ds' is obtained from equation (5).

The relative velocity of the galaxies is obtained from

$$V = \frac{dr}{dt} = \left[\frac{2}{\mu}\left\{E_i - W(r) - \Delta U_1(r) - \Delta U_2(r)\right\}\right]^{\frac{1}{2}} \qquad \dots (20)$$

where

$$\frac{1}{\mu} = \frac{1}{M_1} + \frac{1}{M_2}. \quad \ldots (21)$$

$\Delta U_1(r)$ and $\Delta U_2(r)$ are the changes in the binding energies of the two galaxies. The terms $-\Delta U_1$ and $-\Delta U_2$ take into account the decrease in energy due to dynamical friction. The initial internal energy U_i of a polytropic sphere of index n follows from virial theorem and is given by

$$U_i = -\frac{3}{5-n}\left(\frac{GM^2}{2R}\right). \quad \ldots (22)$$

The binding energy of the two galaxies is obtained from

$$\left.\begin{aligned} U_1 &= U_{i,1} + \Delta U_1 \\ \text{and} \quad & \\ U_2 &= U_{i,2} + \Delta U_2 \end{aligned}\right\} \quad \ldots (23)$$

where ΔU_1 and ΔU_2 are obtained from equation (19).

The translational energy E at any instant due to the relative motion of the two galaxies is

$$E = E_i - \Delta U_1 - \Delta U_2 \quad \ldots (24)$$

where E_i, the initial translational energy, is given by

$$E_i = \tfrac{1}{2}\mu V_i^2 - \frac{GM_1M_2}{R_2}\chi(r_i). \quad \ldots (25)$$

In the impulsive approximation, the change in the translational energy is

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{P^2}{2\mu}\right) \quad \ldots (26)$$

where P is the linear momentum. Since $P = \mu V$, equation (26) reduces to

$$\frac{dE}{dt} = V\frac{dP}{dt} = \mu V f_D \quad \ldots (27)$$

where f_D is the force due to dynamical friction. For a head-on collision of galaxies in the z-direction, this reduces to

$$f_D(z) = \frac{1}{\mu}\left(\frac{dE}{dz}\right). \quad \ldots (28)$$

5. Numerical results and discussion

We have carried out the computations for a head-on collision between two identical galaxies of equal mass and radius and make estimates for the gain in internal energy of M_2 by finding the tidal forces exerted on a number of stars chosen as representative of the galaxy M_2. The stars are chosen on 30 shells of radii differing in steps of $0.02R$ up to a distance of $R_0 = 0.6R$. In each shell of radius a we consider 14 stars, the positions of which are chosen as shown in Table 1. Stars numbered 7 through 14 are chosen on the centres of octants of the sphere and stars numbered 1

Table 1. Positions of stars

Star No.	x'	y'	z'
1	0	0	+a
2	0	0	—a
3	0	+a	0
4	0	—a	0
5	+a	0	0
6	—a	0	0
7	$+a/\sqrt{3}$	$+a/\sqrt{3}$	$+a/\sqrt{3}$
8	$+a/\sqrt{3}$	$-a/\sqrt{3}$	$+a/\sqrt{3}$
9	$-a/\sqrt{3}$	$+a/\sqrt{3}$	$+a/\sqrt{3}$
10	$-a/\sqrt{3}$	$-a/\sqrt{3}$	$+a/\sqrt{3}$
11	$-a/\sqrt{3}$	$-a/\sqrt{3}$	$-a/\sqrt{3}$
12	$+a/\sqrt{3}$	$-a/\sqrt{3}$	$-a/\sqrt{3}$
13	$-a/\sqrt{3}$	$+a/\sqrt{3}$	$-a/\sqrt{3}$
14	$+a/\sqrt{3}$	$+a/\sqrt{3}$	$-a/\sqrt{3}$

through 6 are chosen on the axes of the Cartesian co-ordinates. Because of the symmetry arising from the head-on nature of the collision, the energy gained by all the stars with the same value of z is the same. Hence, the actual computation of energy need be performed only for five stars (we have taken stars 1, 2, 5, 7, 11). It also follows from the symmetry of the collision that $\Delta V'_x = \Delta V'_y$; hence only $\Delta V'_x$ and $\Delta V'_z$ need be tabulated. We choose the units of distance, mass and the gravitational constant so that $R = 1$, $M = 1$, $G = 0.04302$. For $M = 10^{11} M_\odot$, $R =$ 10 kpc, the unit of time is 9.978×10^6 years and the unit of velocity is 1000 km s^{-1}. We reckon time from the instant $z = -3$. Tables 2 and 3 give the values of $\Delta V'_x$ and $\Delta V'_z$ respectively for various positions of M_1 for initial velocity $V_i = 0.700$, while Tables 4 and 5 give these values for $V_i = 1.000$. These correspond to $V_\infty =$ 0.660 and 0.970 respectively. ΔU_1 and ΔU_2 are computed at various separations by using equation (19). The relative velocity of the galaxies is then corrected for dynamical friction by using equation (20) at each instant. Thus the values of $\Delta \mathbf{V}$ given in Tables 4 and 5 take into account the effects of dynamical friction.

The binding energy U of a galaxy obtained with the aid of equation (23) is shown as a function of the separation for various initial values in Figure 2. It may be noted from Figure 2 that the binding energy is practically unchanged until the separation of the centres of M_1 and M_2 becomes as small as 0.5. This underscores the impulsive nature of the collision. U remains negative at all separations for all velocities considered which indicates that the galaxy does not disrupt in any of the collisions. For the case of $V_i = 0.500$, the galaxy M_1 comes to a halt at $z = +0.6$ and then reverses its direction of motion. We have not computed the subsequent motion. The two galaxies will soon merge in this case.

Figures 3a, 3b, 3c show the increments in the binding energy as functions of time as well as of separation. Two vital points may be noted from these histograms :

(a) Most of the increment in the binding energy of the galaxy occurs very near the closest approach. More than 80 per cent of the increment in binding energy takes place when the centres of the two galaxies are separated by a distance less than $0.4R$,

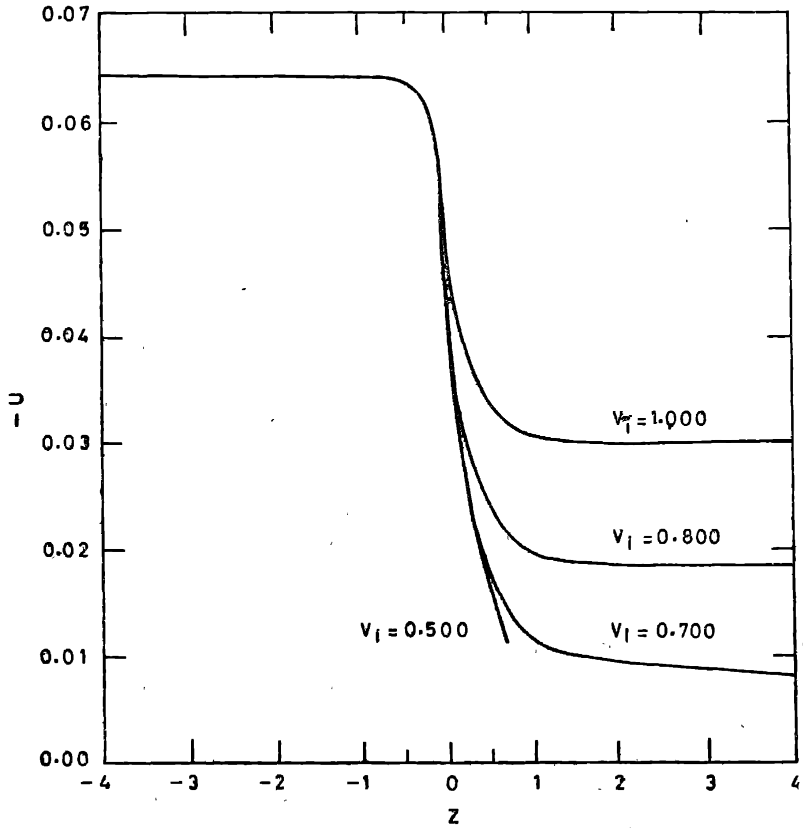

Figure 2. Binding energy of a galaxy during collision (dynamical friction taken into account).

and more than 90 per cent when the separation is less than 0.8 R. It is of interest to compare this result with an inference that can be drawn from the pioneering work of Spitzer (1958). It can be deduced from Spitzer's theory that in a distant encounter about 90 per cent of the energy gain takes place when the separation of the galaxies is less than 1.5 times the distance of closset approach.

(b) The change in the binding energy is quite asymmetric with respect to the distance of closest approach. The increment in the sceond half of the encounter is roughly double that of the first half. The simplifying assumption earlier made by Alladin *et al.* (1975) and Sastry & Alladin (1977), namely,

$$(\Delta U_1)_p + (\Delta U_2)_p = \tfrac{1}{2}(\Delta U_1 + \Delta U_2) \qquad \text{... (29)}$$

should therefore be modified in a head-on collision to

$$(\Delta U_1)_p + (\Delta U_2)_p \approx \tfrac{1}{3}(\Delta U_1 + \Delta U_2) \qquad \text{... (30)}$$

where $(\Delta U_1)_p + (\Delta U_2)_p$ represents the increment in binding energy of the two galaxies during the first half of the encounter and $(\Delta U_1 + \Delta U_2)$, the total increment for the entire encounter. The calculations show that there is asymmetry in energy transfer in a head-on collision even if dynamical friction is neglected. The asymmetry in energy transfer decreases as the distance of closest approach of the two galaxies increases. It can be deduced from the analysis carried out by Spitzer

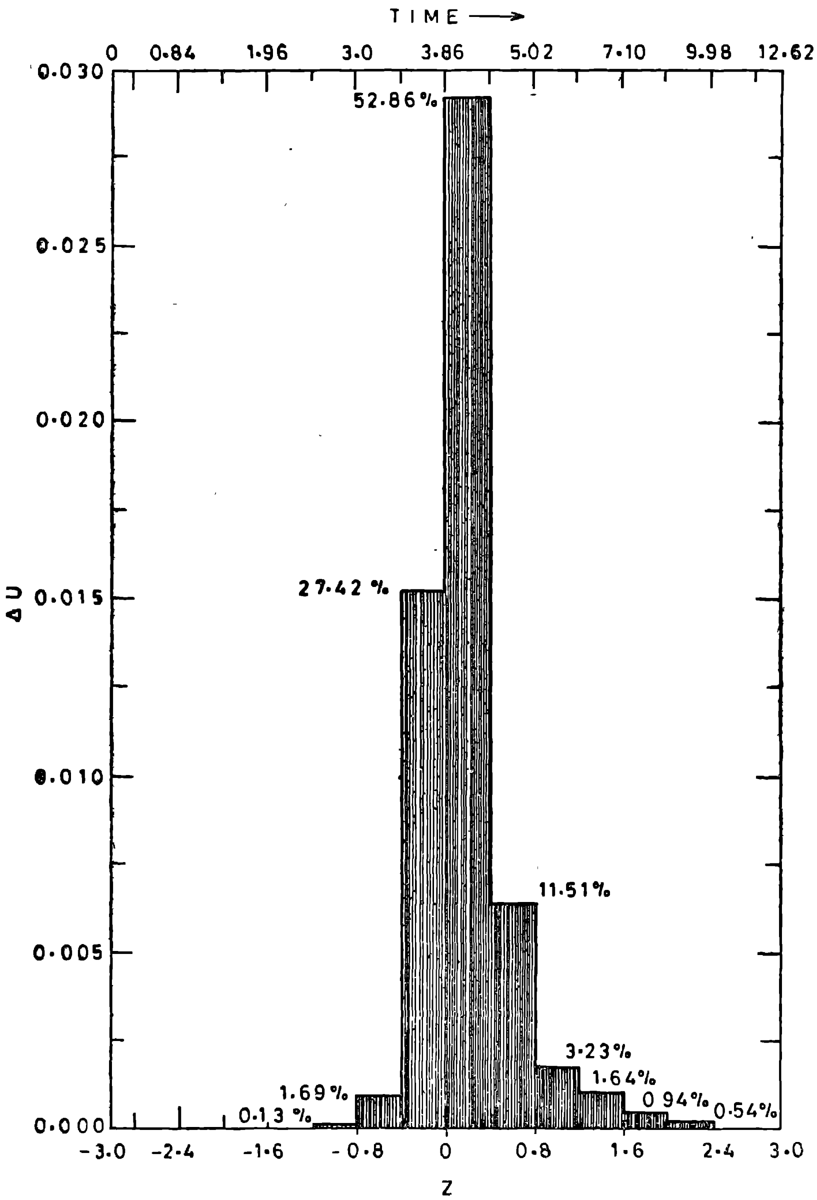

Figure 3a. Asymmetry in change in binding energy along the orbit for $Vi = 0.700$.

(1958) that for distant encounters equation (29) is valid i.e. there is no asymmetry. Lauberts (1974) derived the fractional change in the internal energy of a galaxy as a function of separation for off-centre collisions. His results show that there is a relatively small disturbance in shape until the galaxies are at their closest approach and after this galaxies deform substantially. The computer simulation of Miller & Smith (1980) performed for very close collisions by taking 50,000 stars in each galaxy also indicates considerable asymmetry in the energy change. The post-collision effect is much more than the pre-collision effect.

The translational energy of the pair of galaxies obtained from equation (24) is shown as a function of the separation for various initial velocities in Figure 4. It

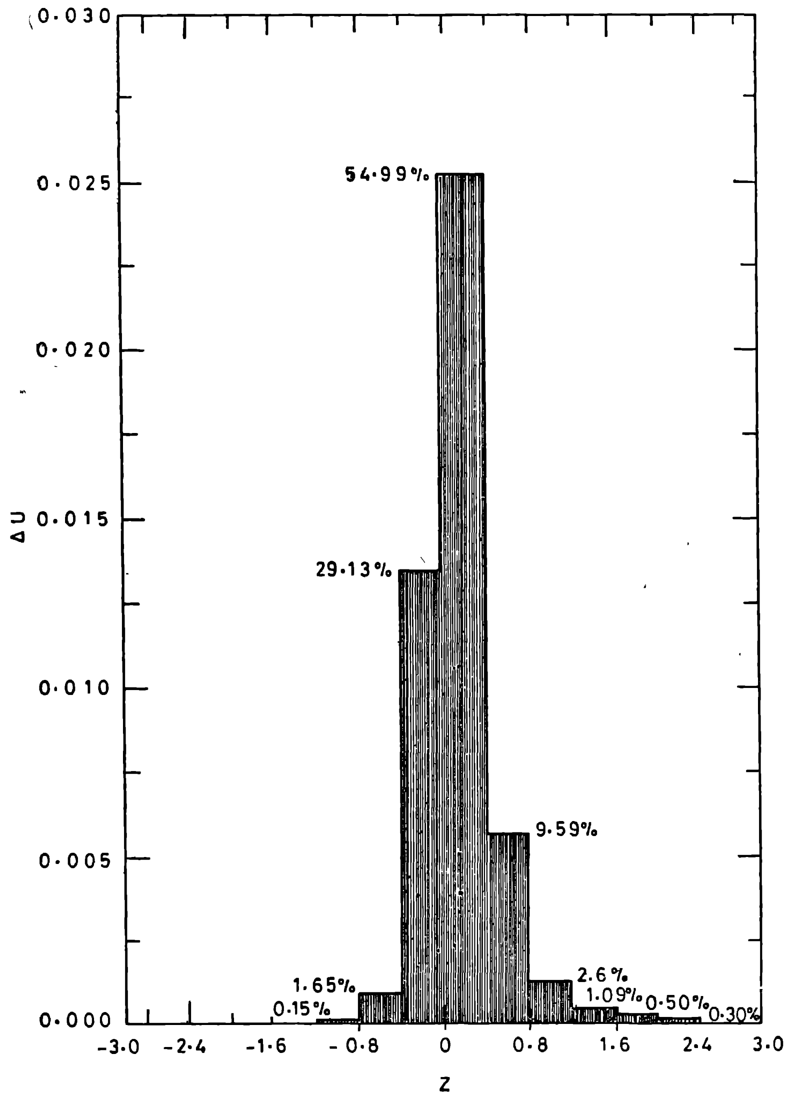

Figure 3b. Asymmetry in change in binding energy along the orbit for $V_i = 0.800$.

may be noted from Figure 4 that the translational energy becomes negative in the cases $V_i = 0.700$ and $V_i = 0.500$ implying that tidal capture takes place. In the case of $V_i = 0.700$, E becomes slightly negative after the collision. The calculations show that M_1 will come to a halt when separation becomes $10R$. The direction of motion will then be reversed, the two galaxies would perform damped oscillatory motion about their common centre of mass and will finally merge. In the case of $V_i = 0.500$ the loss of translational energy is considerable and would therefore result in a rapid merger. With increasing initial relative velocity V_i the loss of translational energy rapidly decreases. The consequent changes in orbital velocity are shown in Figure 5 which also shows the vital role played by dynamical friction

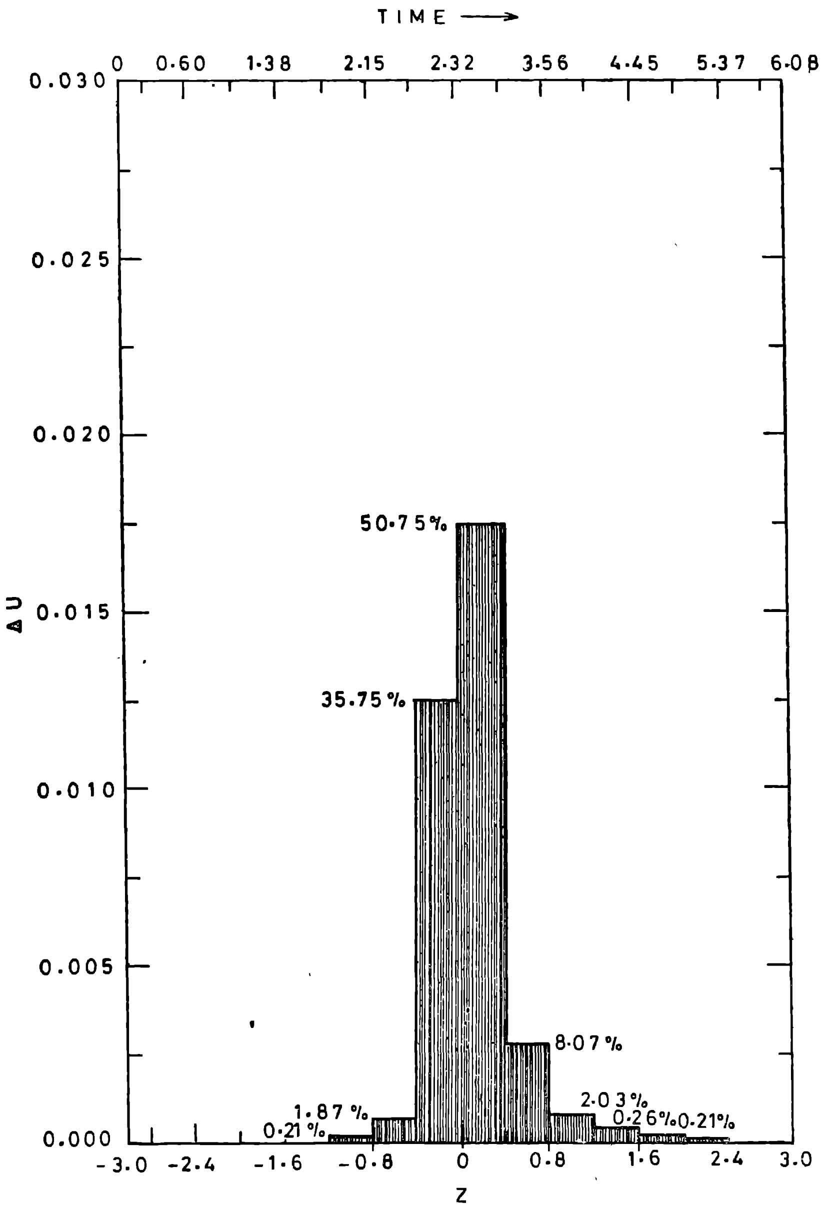

Figure 3c. Asymmetry in change in binding energy along the orbit for $V_i = 1.000$.

in reducing the relative velocity of galaxies. The asymmetry in energy transfer about the distance of closest approach is also evident from these curves. Tables 6 and 7 give the binding and translational energies and the velocity as a function of separation of the two galaxies and time.

The decelerating force f_D due to a dynamical friction obtained from equation (28) is shown as a function of separation of the galaxies in Figure 6 for $V_i = 1.000$ and $V_i = 0.700$. The curves show that f_D rises steeply as the distance of closest approach draws near and the fall in f_D is less steep when the galaxies are receding. This also brings into prominence the asymmetry in energy transfer about the position of closest approach.

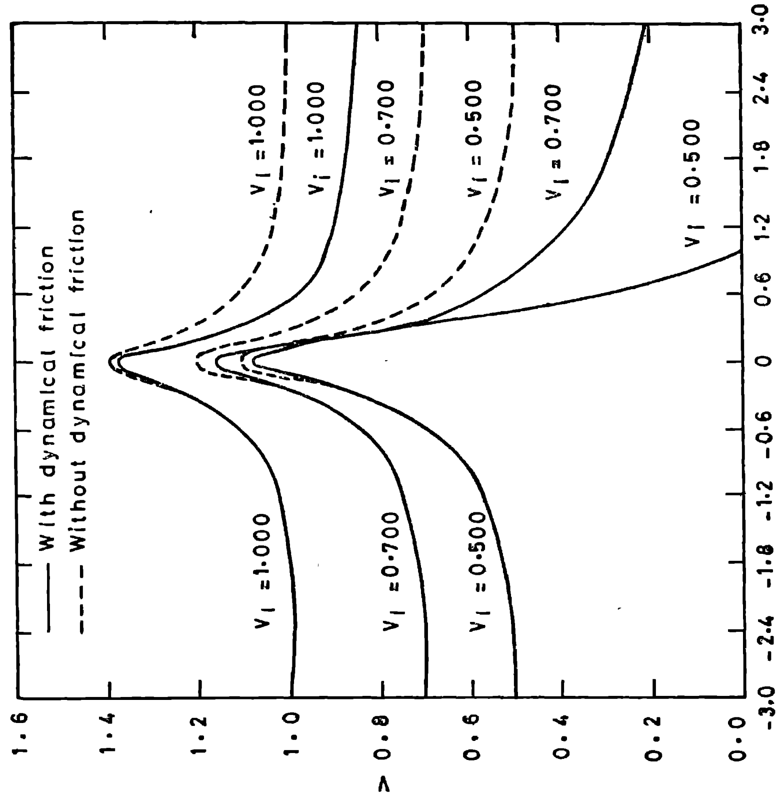

Figure 5. Relative velocity of colliding galaxies.

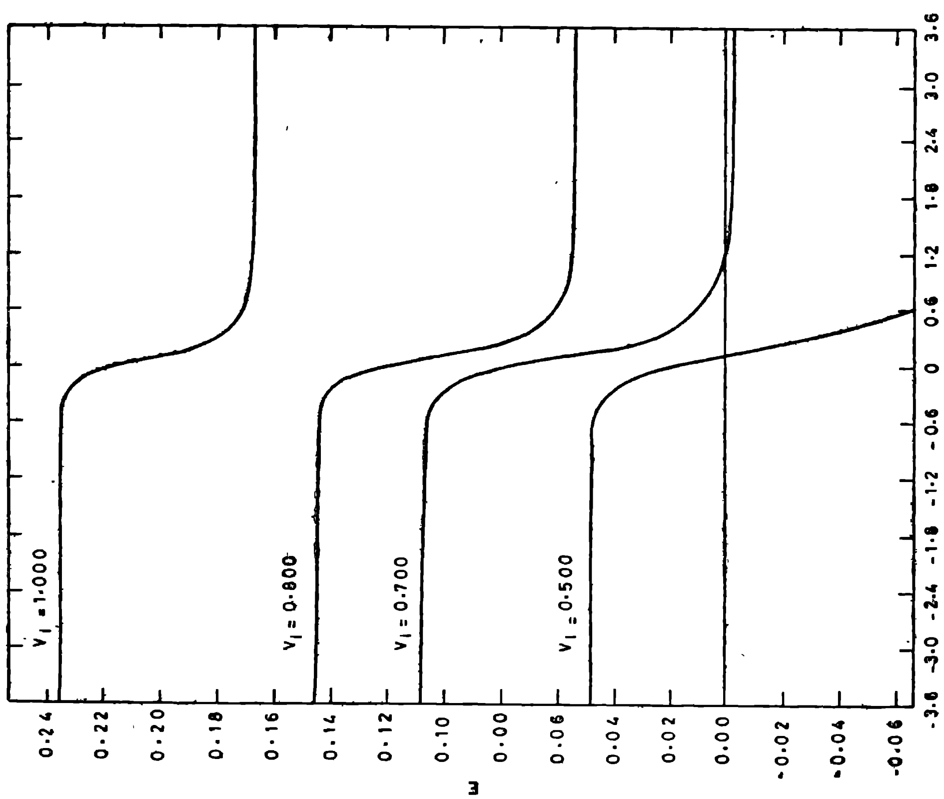

Figure 4. Translational energy of galaxies.

Table 2. $\Delta V_x' \times 1000$ for $V_i = 0.700$

Star position	Star No.						
		t 1.410	2.740	3.860	5.460	8.440	12.620
		z —2.0	—1.0	0.0	1.0	2.0	3.0
		V 0.720	0.778	1.198	0.437	0.282	0.212
0.00		0.00	0.00	0.00	0.00	0.00	0.00
0.02							
	1	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—0.08	—0.51	—58.0	—120.0	—121.0	—121.0
	7	—0.05	—0.29	—30.6	—69.8	—70.3	—70.5
	11	0.05	0.30	36.8	69.4	69.9	70.0
0.06							
	1	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—0.25	—1.53	—146.0	—302.0	—305.0	—306.0
	7	—0.14	—0.82	—67.3	—188.0	—190.0	—190.0
	11	0.15	0.96	113.0	185.0	186.0	186.0
0.10							
	1	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—0.42	—2.54	—184.0	—382.0	—386.0	—388.0
	7	—0.22	—1.30	—77.3	—259.0	—262.0	—262.0
	11	0.26	1.46	168.0	251.0	253.0	254.0
0.14							
	1	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—0.58	—3.53	—188.0	—391.0	—397.0	—399.0
	7	—0.31	—1.85	—74.0	—287.0	—291.0	—292.0
	11	0.37	2.45	195.0	275.0	278.0	279.0
0.20							
	1	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—0.83	—4.97	—169.0	—354.0	—363.0	—365.0
	7	—0.42	—2.30	—64.1	—286.0	—292.0	—294.0
	11	0.56	3.75	199.0	268.0	272.0	274.0
0.30							
	1	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—1.23	—7.17	—132.0	—279.0	—291.0	—295.0
	7	—0.58	—3.03	—48.4	—247.0	—258.0	—261.0
	11	0.90	6.23	173.0	226.0	231.0	233.0
0.40							
	1	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—1.61	—9.07	—106.0	—223.0	—239.0	—243.0
	7	—0.72	—3.56	—38.1	—208.0	—224.0	—227.0
	11	1.27	9.06	145.0	186.0	192.0	194.0
0.50							
	1	0.00	0.00	0.00	0.00	0.00	0 00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—1.97	—10.6	—87.6	—183.0	—201.0	—207.0
	7	—0.83	—3.93	—31.3	—174.0	—196.0	—201.0
	11	1.69	12.1	123.0	156.0	163.0	165.0
0.60							
	1	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.00	0.00	0.00	0.00	0.00	0.00
	5	—2.29	—11.9	—74.6	—154.0	—174.0	—180.0
	7	—0.92	—4.1	—26.5	—148.0	—175.0	—181.0
	11	2.15	15.2	106.0	134.0	140.0	143.0

Table 3. $\Delta V_z^i \times 1000$ for $V_i = 0.700$

Star position	Star No.							
		t	1.410	2.740	3.860	5.460	8.440	12.620
		z	—2.0	—1.0	0.0	1.0	2.0	3.0
		V	0.720	0.778	1.198	0.437	0.282	0.212
0.00			0.00	0.00	—127.5	8.25	8.25	8.25
0.02								
	1		0.17	1.00	—119.0	19.0	20.9	21.3
	2		—0.17	—1.05	—127.0	—2.49	—4.23	—4.71
	5		0.00	0.01	—123.0	7.94	7.91	7.91
	7		0.10	5.92	—121.0	14.3	15.4	15.7
	11		—0.10	—0.59	—125.0	17.3	0.71	0.43
0.06								
	1		0.48	2.89	—81.9	40.8	46.5	47.8
	2		—0.52	—3.29	—105.0	—23.5	—28.5	—29.9
	5		0.01	0.10	—90.5	5.66	5.40	5.45
	7		0.29	17.7	—85.1	24.1	27.1	28.0
	11		—0.29	—1.77	—98.1	—10 93	—14.0	—14.8
0.10								
	1		0.79	4.60	—33.0	—62.2	72.2	74.8
	2		—0.90	—5.74	—70.9	—42.4	—50.4	—52.7
	5		0.03	0.27	43.9	2.01	1.50	1.43
	7		0.48	2.94	—36.8	29.7	34.7	36.1
	11		—0.48	—2.93	—55.6	—21.4	—26.4	—27.8
0.14								
	1		1.08	6.24	12.9	82.7	97.5	101.0
	2		—1.29	—8.44	36.9	—58.31	—69.0	—72.1
	5		0.05	0.53	1.32	—2.09	—3.07	—3.22
	7		0.68	4.09	9.06	31.1	38.2	40.1
	11		—0.68	—4.08	—12.9	28.7	35.8	37.8
0.20								
	1		1.49	8.43	66.6	107.0	130.0	136.0
	2		—1.92	—12.0	3.14	—76.3	—90.7	—94.0
	5		0.10	1.07	55.5	—8.04	—10.0	—10.3
	7		0.96	5.78	63.1	27.4	37.2	39.9
	11		—0.96	—5.71	39.7	— 35.0	—45.0	—47.0
0.30								
	1		2.11	11.6	122.0	125.0	165.0	174.4
	2		—3.10	—22.5	48.3	—96.6	—111.0	—117.0
	5		0.23	2.32	113.0	—16.1	—20.4	—21.1
	7		1.43	8.46	120.0	13.2	27.1	31.2
	11		—1.41	—8.12	98.0	—38.0	—52.8	—56.9
0.40								
	1		2.67	14.3	155.0	124.0	188.0	220.0
	2		—4.46	—35.3	80.7	—94.0	—118.0	—125.9
	5		0.41	3.93	148.0	—21.9	—29.1	—30.2
	7		1.88	10.9	153.0	—3.01	13.8	19.2
	11		—1.85	—9.93	135.0	—37.7	—56.5	—61.9
0.50								
	1		3.17	16.5	176.0	95.1	191.0	208.0
	2		—6.05	—53.4	107.0	—89.3	—117.0	—126.0
	5		0.63	5.80	170.0	—25.6	—36.2	—37.9
	7		2.32	13.2	174.0	—17.6	—0.64	—70.7
	11		—2.24	—10.0	160.0	—35.9	—59.5	—65.1
0.60								
	1		3.62	18.5	190.0	44.1	190.0	213.0
	2		7.92	80.5	129.0	—81.6	—112.0	—123.0
	5		0.88	7.81	186.0	—27.9	—42.1	—44.6
	7		2.73	15.3	189.0	—29.3	—11.6	—4.22
	11		—2.58	—10.9	178.0	—33.6	—59.7	—67.6

Table 4. $\Delta V_x^{\prime} \times 1000$ for $V_i = 1.000$

Star position	Star No.							
		t	0.980	1.960	2.820	3.780	4.920	6.080
		z	−2.00	−1.00	0.00	1.00	2.00	3.00
		V	1.014	1.056	1.399	0.921	0.868	0.851
0.00			0.00	0.00	0.00	0.00	0.00	0.00
0.02								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−0.06	−0.37	−48.8	−98.6	−99.0	−99.4
	7		−0.03	−0.20	−25.8	−57.4	−57.6	−57.6
	11		0.03	0.21	30.9	57.4	57.4	57.4
0.06								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−0.17	−1.00	−123.0	−248.0	−249.0	−249.0
	7		−0.09	−0.59	−56.4	−153.0	−154.0	−154.0
	11		0.10	0.69	94.7	152.0	153.0	153.0
0.10								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−0.29	−1.83	−154.0	−311.0	−313.0	−313.0
	7		−0.15	−0.94	−64.7	−209.0	−210.0	−210.0
	11		0.18	1.20	140.0	206.0	207.0	207.0
0.14								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−0.41	−2.55	−156.0	−316.0	−318.0	−319.0
	7		−0.21	−1.25	−62.1	−229.0	−230.0	−231.0
	11		0.26	1.77	162.0	225.0	226.0	227.0
0.20								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−0.58	−3.58	−139.0	−282.0	−285.0	−286.0
	7		−0.29	−1.65	−53.0	−222.0	−225.0	−225.0
	11		0.39	2.70	164.0	217.0	219.0	219.0
0.30								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−0.86	−5.17	−108.0	−216.0	−221.0	−222.0
	7		−0.40	−2.18	−39.6	−184.0	−188.0	−189.0
	11		0.63	4.50	141.0	180.0	182.0	182.0
0.40								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−1.13	−6.54	−85.2	−169.0	−175.0	−176.0
	7		−0.50	−2.56	−30.8	−148.0	−155.0	−156.0
	11		0.89	6.54	116.0	146.0	148.0	149.0
0.50								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−1.38	−7.67	−69.7	−136.0	−143.0	−145.0
	7		−0.58	−2.82	−25.1	−121.0	−129.0	−131.0
	11		1.19	7.84	134.0	121.0	123.0	124.0
0.60								
	1		0.00	0.00	0.00	0.00	0.00	0.00
	2		0.00	0.00	0.00	0.00	0.00	0.00
	5		−1.61	−8.56	−58.8	−113.0	−121.0	−123.0
	7		−0.65	−2.99	−21.1	−99.4	−110.0	−112.0
	11		1.51	11.0	92.9	102.0	105.0	106.0

Table 5. $\Delta V'_z \times 1000$ for $V_i = 1.000$

Star position	Star No.							
		t	0.980	1.960	2.820	3.780	4.920	6.080
		z	−2.00	−1.00	0.00	1.00	2.00	3.00
		V	1.014	1.056	1.399	0.921	0.868	0.851
0.00			0.00	0.00	−107.2	3.07	3.07	3.07
0.02								
	1		0.11	0.72	−101.0	7.64	8.39	8.52
	2		−0.12	−0.75	−106.0	−1.55	−2.25	−2.39
	5		0.00	0.00	−103.0	2.97	2.96	2.96
	7		0.06	0.42	−102.0	5.75	6.16	6.24
	11		−0.06	−0.42	−105.0	−0.24	−0.24	−0.25
0.06								
	1		0.34	2.08	−71.4	16.5	18.8	19.3
	2		−0.36	−2.37	−84.5	−10.8	−12.8	−13.2
	5		0.00	0.07	−76.1	2.24	2.14	2.14
	7		0.20	1.47	−73.1	9.87	11.1	11.3
	11		−0.20	−1.27	−80.4	−5.05	−6.29	−6.53
0.10								
	1		0.55	3.34	−31.2	−24.2	28.6	29.3
	2		−0.63	−4.14	−52.2	−191.0	22.4	23.1
	5		0.01	0.19	37.1	1.08	0.87	0.85
	7		0.34	2.12	−33.2	11.9	14.0	14.4
	11		−0.34	−2.12	−43.7	−9.03	−11.1	−11.5
0.14								
	1		0.76	4.49	6.69	31.2	37.2	38.3
	2		−0.91	−6.09	−20.7	−26.0	30.3	31.2
	5		0.03	0.38	0.47	−0.11	−0.51	−0.56
	7		0.47	2.95	4.65	12.1	15.0	15.6
	11		−0.47	−2.94	−7.39	−11.3	−14.2	−14.8
0.20								
	1		1.05	6.08	51.0	37.5	46.8	48.4
	2		−1.35	−9.41	16.7	−33.2	−39.0	−40.2
	5		0.07	0.77	45.2	−1.66	−2.47	−2.56
	7		0.67	4.17	49.2	9.87	13.8	14.6
	11		−0.67	−4.12	36.8	−12.40	−16.4	−17.2
0.30								
	1		1.49	8.36	97.3	34.0	50.3	52.0
	2		−2.18	−16.3	58.7	−37.1	−45.1	−46.8
	5		0.16	1.68	92.8	−3.32	−5.07	−5.27
	7		1.00	6.10	95.0	3.27	−8.89	10.1
	11		−1.00	−5.86	85.2	−10.9	−16.8	−18.0
0.40								
	1		1.88	10.3	123.0	20.7	46.7	50.4
	2		−3.14	−25.6	86.9	−34.7	−44.4	−46.6
	5		0.26	2.59	116.0	−3.9	−6.62	−6.92
	7		1.26	7.54	118.0	−2.04	4.54	6.00
	11		−1.24	−6.92	109.0	−8.72	15.9	17.5
0.50								
	1		2.23	11.9	140.0	−2.20	37.6	42.6
	2		−4.26	−38.7	108.0	−9.19	−40.4	−43.0
	5		4.43	4.19	38.2	−3.94	−8.28	−8.80
	7		1.63	9.53	140.0	−8.46	−1.20	0.64
	11		−1.58	−7.84	134.0	−5.26	−14.4	−16.3
0.60								
	1		2.55	13.3	−151.0	28.0	30.9	34.5
	2		−5.58	−58.4	125.0	−22.9	−35.3	38.3
	5		0.62	5.65	150.0	−3.43	−9.25	−9.98
	7		1.92	11.0	151.0	−12.1	−5.13	−3.00
	11		−1.82	−7.82	147.0	−2.62	−3.11	−15.4

Table 6. Binding and translational energies for V_i = 0.700

z	t	$\Delta U_1 = \Delta U_2$	U	$-\Delta E$	E	V
—3.00	0.00	0.00	—0.645(—1)*	0.00	0.108	0.700
—2.60	0.56	0.432(—7)	—0.645(—1)	0.865(—7)	0.108	0.706
—2.20	1.13	0.288(—6)	—0.645(—1)	0.576(—6)	0.108	0.715
—1.80	1.68	0.122(—5)	—0.645(—1)	0.244(—5)	0.108	0.726
—1.40	2.21	0.492(—5)	—0.645(—1)	0.984(—5)	0.108	0.745
—1.00	2.74	0.243(— 4)	—0.645(—1)	0.486(—4)	0.108	0.778
—0.80	3.01	0.672(—4)	—0.644(—1)	0.134(—3)	0.108	0.804
—0.60	3.25	0.249(—3)	—0.643(—1)	0.499(—3)	0.108	0.848
—0.40	3.48	0.101(—2)	—0.635(—1)	0.201(—2)	0.106	0.924
—0.20	3.68	0.395(—2)	—0.606(—1)	0.789(—2)	0.100	1.069
0.00	3.86	0.162(—1)	—0.483(—1)	0.324(—1)	0.758(—1)	1.198
0.20	4.05	0.359(—1)	—0.286(—1)	0.719(—1)	0.363(—1)	1.010
0.40	4.30	0.455(—1)	—0.190(—1)	0.911(—1)	0.171(—1)	0.754
0.60	4.63	0.496(—1)	—0.149(—1)	0.991(—1)	0.904(—2)	0.596
0 80	5.02	0.517(—1)	—0.126(—1)	0.103	0.486(—2)	0.501
1.00	5.46	0.529(—1)	—0.116(—1)	0.106	0.238(—2)	0.437
1.20	5.96	0.537(—1)	—0.108(—1)	0.107	0.800(—3)	0.391
1.40	6.50	0.542(—1)	—0.103(—1)	0.108	—0.260(—3)	0.355
1.80	7.77	0.549(—1)	—0.100(—1)	0.110	—0.158(—2)	0.302
2.20	9.20	0.553(—1)	—0.920(—1)	0.110	—0.200(—2)	0.261
2.60	10.82	0.555(—1)	—0.900(—1)	0.111	—0.300(—2)	0.233
3.00	12.62	0.557(—1)	—0.900(—1)	0.111	—0.300(—2)	0.212

*The numbers in brackets denote powers of ten, e.g., —0.645(—1) means —0.645 × 10^{-1}.

Table 7. Binding and translational energies for V_i = 1.000

z	t	$\Delta U_1 = \Delta U_2$	U	$-\Delta E$	E	V
—3.00	0.00	0.00	—0.645(—1)	0.00	0.236	1.000
—2.60	0.40	0.213(—8)	—0.645(—1)	0.426(—8)	0.236	1.004
—2.20	0.78	0.143(—7)	—0.645(—1)	0.286(—7)	0.236	1.010
—1.80	1.19	0.611(—6)	—0.645(—1)	0.122(—5)	0.236	1.019
—1.40	1.58	0.250(—5)	—0.645(—1)	0.500(—5)	0.236	1.032
—1.00	1.96	0.127(—4)	—0.645(—1)	0.254(—4)	0.236	1.056
—0.80	2.15	0.358(—4)	—0.645(—1)	0.716(—4)	0.236	1.076
—0.60	2.33	0.138(—3)	—0.644(—1)	0.276(—3)	0.236	1.108
—0.40	2.48	0.583(—3)	—0.638(—1)	0.118(—2)	0.235	1.168
—0.20	2.68	0.246(—2)	—0 621(—1)	0 493(—2)	0.233	1.287
0.00	2.82	0.112(—1)	—0.515(—1)	0.222(—1)	0.213	1.399
0.20	2.98	0.248(—1)	—0.397(—1)	0.497(—1)	0.186	1.254
0.40	3.16	0.305(—1)	—0.340(—1)	0.611(—1)	0.175	1.080
0.60	3.36	0.325(—1)	—0.321(—1)	0.649(—1)	0.171	0.992
0.80	3.56	0.333(—1)	—0.312(—1)	0.666(—1)	0.169	0.947
1.00	3.78	0.337(—1)	—0.308(—1)	0.675(—1)	0.168	0.921
1.20	4.00	0.340(—1)	—0.305(—1)	0.680(—1)	0.168	0.903
1.40	4.22	0.342(—1)	—0.304(—1)	0.683(—1)	0.167	0.891
1.80	4.68	0.343(—1)	—0.302(—1)	0.687(—1)	0.167	0.874
2.20	5.14	0.344(—1)	—0.301(—1)	0.688(—1)	0.167	0.863
2.60	5.60	0.345(—1)	—0.300(—1)	0.690(—1)	0.167	0.856
3.00	6.08	0.345(—1)	—0.300(—1)	0.690(—1)	0.167	0.851

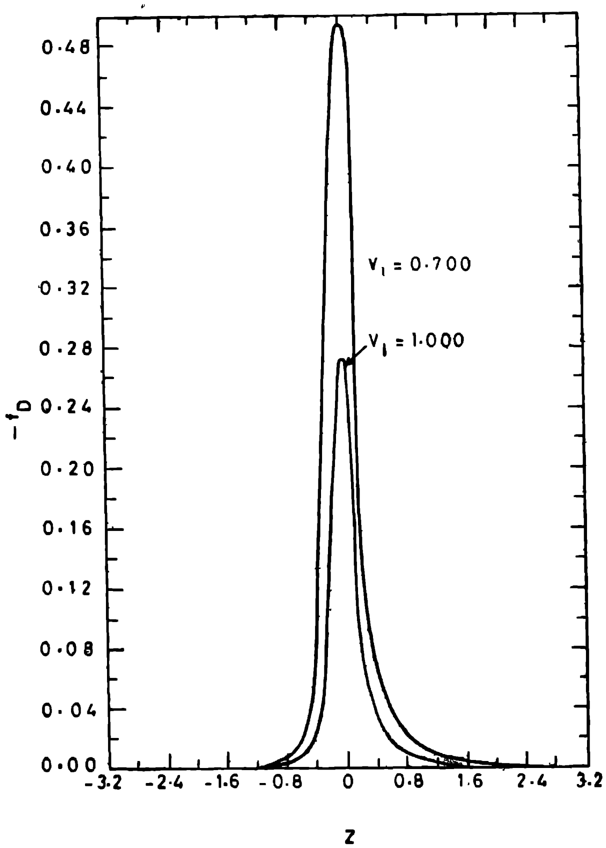

Figure 6. Deceleration due to dynamical friction.

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