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Seismic Probing of the Solar Core

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Abstract. Only a small fraction of the solar p-modes have frequencies with any measurable sensitivity to the core structure. Nevertheless an accurate seismic sounding with such modes alone is possible. This requires a high-precision measurement of mode frequencies and a proper method of inversion. The important aspects here are elimination of the uncertainties concerning the outermost layers and making best use of *a priori* information about structural variables. Current data allow us to achieve an accuracy in pressure and density determination at the level of 1 percent in the bulk of the core. Results of helioseismic inversions support the standard picture of solar evolution.

Key words: Sun: interior – Sun: oscillations

1. Introduction

High energy neutrino flux used to be regarded as an excellent thermometer for the solar center. Today the prevailing view is that the solution of the solar neutrino problem requires modification in neutrino physics. The most popular modification is the MSW theory of neutrino conversion (e.g. Bahcall, 1989). In its simplest version – obtained by assuming that the electron neutrinos are converted into only one of the other neutrino species – it contains two free parameters. In principle, available experimental data may be used to estimate the central temperature simultaneously with these two parameters, but a poor accuracy makes such determination uninteresting.

Gravity mode frequencies exhibit strong sensitivity to core structure. Unfortunately, none of the many announcements of detection of the solar gravity modes have been confirmed by independent studies. Prospects for detecting such modes seem to me not very good, but I would like to stress that this highly personal view of a theorist should not discourage observers from continuing searches.

In any case, for the time being, the only tools for probing the Sun's core are p-modes. Most of them are practically insensitive to the core structure, but there is a significant number of modes for which fractional changes in sound speed in the core, at the level of 10^{-2}, causes a measurable change in frequency. I will quantify this statement further in Section 3. Even in the best cases the effect is small and, therefore, probing the core requires very high precision in frequency measurements and care in the inversion procedure.

In this review I focus on methodology of the inversion. I will survey basic underlying assumptions, uncertainties involved in the process and accuracy of parameter determination. Significance of accurate sounding of the core for solar neutrino problem and for testing physical ingredients of the stellar evolution theory will be discussed at the end.

2. The inverse problem

Common assumptions made in seismic probings of the Sun's deep interior structure include hydrostatic equilibrium, adiabaticity of oscillations, and, in most cases, validity of linearization about a reference model.

The assumption of hydrostatic equilibrium is partially testable by means of measuring the fine structure induced by rotation and magnetic fields. The relative frequency shifts in centroids due to centrifugal distortion is somewhat below 10^{-5} and thus slightly below the current level of measurement accuracy. The effect, however, should not be forgotten. The magnetic field varying with the solar cycle induces larger shifts. This effect may and should be removed from data used for a structural inversion. Small scale velocity fields are not directly detectable. We eliminate the effects of such fields in outer layers, where we know that they are present at a significant level (average Mach number $\sim 10^{-3}$), in the way I will soon describe.

The assumption of adiabaticity of oscillations is certainly not justified at the level of accuracy of the observations. Fortunately, since nonadiabatic effects are localized in the outermost layers, they may be eliminated in the same manner as the effects of small scale velocity fields.

The linearization about a reference model is adopted in most of the inversions aimed at determining structure of the solar core and I believe it is necessary for accurate probing. Methods based on asymptotic approximation (Brodski and Vorontsov 1988; Vorontsov and Shibahashi 1991) do not require reference models. This is an advantage but I am skeptical about prospects of improving this approximation so that it would be applicable in the core.

With these assumptions we may use of the variational principle to connect unknown differences , Δ, in structural parameters between the Sun and a model to known differences in frequencies (e.g. Gough and Thompson 1991). An *ad hoc* term of the form $F(\nu)/I$, where ν is the frequency an I is calculated moment of inertia, must be added to account for large departures from simplifying assumptions near the surface of the sun. Such form follows from the fact that there the radial eigenfunctions are nearly l-indepedent. The resulting formula may be written in the following form,

$$\left(\frac{\Delta\nu}{\nu}\right)_j = \int \mathcal{K}_{f,j}\frac{\Delta f}{f}dx + \int \mathcal{K}_{\Gamma,j}\frac{\Delta\Gamma_1}{\Gamma_1}dx + \frac{F(\nu)}{I_j}, \tag{1}$$

where $j \equiv (l, n)$ identifies the mode, $f(x)$ is a structural function, and $x = r/R$. There is freedom in choosing f. It could be density, pressure, or any combination of these parameters and their derivatives. All such functions are connected through the linearized mechanical equilibrium condition. Our goal is to determine three functions: $f(x)$, $\Gamma_1(x)$ and $F(\nu)$ using Eq.(1) for each of over 2000 p-modes with individually measured frequencies. It is assumed here that solar mass and radius are known exactly, while in fact there is an uncertainty. It is easy to allow for it.

Additional assumptions, that have been include, are $\Gamma_1 = \Gamma_1(p, \rho, Y)$ and $Y = Y(1)$ in the envelope, and $\Delta\Gamma_1 = 0$ in the radiative interior. They allow us to replace unknown function, $\Delta\Gamma_1$, with unknown number $\Delta Y(1)$. This is an advantage because the demand on information that we want to extract from a given set of data is reduced. However, there is an uncertainty in Γ_1 and these simplifying assumptions introduce an error in determination of $f(x)$. Both strategies have been applied. I believe that for studies of the core structure it is better to make use of these additional assumptions but I have no proof.

An inversion of Eq.(1) yields directly p, ρ in the whole interior. With the above made assumptions we determine also Y in the outer layers, from the photosphere to the base of the convective envelope. Making use of the equation of state we can also infer $T(x)$ in these part of the Sun. Below the convective zone, due to gravitational settling and, deeper down, nuclear burning, Y must be regarded as a function of x. Without introducing new constraints we cannot determine this function because in these deep layers because $\Gamma_1 \approx 5/3$ does not have sufficient sensitivity to Y. These additional constraints follow from the thermal equilibrium condition and involve data on the energy generation and opacity.

In the further discussion I will follow the methodology of inversion developed by our group in Warsaw (Dziembowski, Pamyatnykh and Sienkiewicz, 1990, DPS) with improvements introduced in collaboration with Phil Goode (Dziembowski et al. 1994, DGPSa). We determine directly $\Delta u/u$, where $u = P/\rho$ is the squared isothermal sound speed of the sound, and $\Delta Y(1)$. Subsequently, we use a linearized mechanical equilibrium condition to infer the remaining parameters. Däppen et al. (1991) developed an alternative method based on the same assumptions. Antia & Basu (1994) do not use the assumptions concerning Γ_1. The results of these inversions are similar.

I will not discuss direct inference concerning temperature in the core and the neutrino flux. This has been done by Gough & Kosovichev (1990), DPS, Antia & Chitre (1995) and Shibahashi & Takata (1996). In some cases very discrepant conclusions have been reached.

3. Kernels

The probing potential of various modes is best revealed in plots of the kernels for structural variables. I show in Fig. 1 three examples of kernels, $\mathcal{K}_u$, for variable $u(x)$ in the radiative interior. The definition of the solar core is to some extent arbitrary. In the figure I marked the region where 95 percent of the solar luminosity is produced and I will call this region the core. It extends up to $x = 0.225$. The region inside $x = 0.1$ I will call the inner core. A comparison of the kernels for the $l = 0$, p_{22} and the

$l = 7$, p_{19} modes, which have very similar frequencies, illustrates properties that are well understood in terms of p-mode asymptotics. The latter mode has its lower turning point at $x = 0.23$ and as the kernel shows the frequency has essentially no sensitivity to the sound speed in the core – in contrast to the $l = 0$ mode. One may also see that the two kernels gradually become more alike when we move away from the turning point. In the asymptotic approximation we have $\mathcal{K} \approx \mathcal{E} > 0$, where $\mathcal{E}$ is the kernel for the mode kinetic energy. We clearly see that this approximation is quite inadequate in the core for the case of the $l = 0$ mode. A failure of the asymptotics for the $l = 7$, p_8 has a more interesting implication. As seen in Fig. 1 the kernel for this mode is quite large even well below the turning point $x = 0.37$. This is a consequence of a nonlocal nature of the kernel in the nonasymptotic treatment. The change of the turning point affects frequencies indirectly through the implied change of the pressure above the turning point.

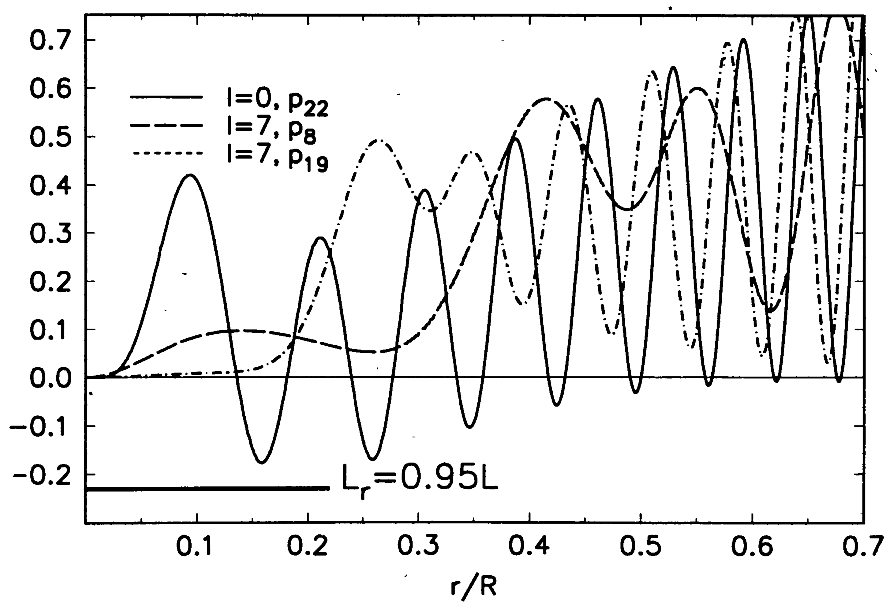

Figure 1. Kernels $\mathcal{K}_u$ for three p-modes. Measured frequencies of the two high order modes are 3168 and 3143 μHz for $l = 0$ and 7, respectively. The p_8 mode frequency is 1628 μHz.

The three modes used in Fig. 1 belong to the set used by Dziembowski et al. (1995, DGPSb) which contains modes whose frequencies were measured at Big Bear Solar Observatory (Libbrecht, Woodard & Kaufman 1990) and low degree modes whose frequencies were obtained by BISON network (Elsworth et al. 1994). The same set has been used in all results of inversions quoted here. The whole combined set contains 2254 frequencies for $l \leq 150$ modes. Only a small fraction of these frequencies is sensitive in a significant way to the speed of sound in the inner core. The condition that a 1 percent change of u in the inner causes a frequency change greater than the frequency error, σ, which we may write, as

$$0.01 \int_0^{0.1} \mathcal{K}_u dx \geq \frac{\sigma}{\nu}, \tag{2}$$

is satisfied by 33 modes in the $l = 0 - 2$ range and by 20 modes in the $l = 5 - 13$ range (5 at $l = 7$). In the former group the contribution to K_u is totally nonlocal. The fact that none of the modes with $l = 3$ and 4 satisfies criterion (2) is in part due to cancellation of local and nonlocal contributions and in part due to relatively large errors. Still the number of modes satisfying this criterion seems large enough for accurate probing of the core including its inner part. The problem lies in eliminating effects of uncertainties concerning rest of the Sun which have an overwhelming influence on p-mode frequencies. For this task we have at our disposal frequencies of all 2254 modes.

4. Method of solution

Even with this large, but still finite number of frequency data we cannot solve the problem of determination of functions, without an additional constraint. This is an assumption that $\Delta u/u(x)$ and $F(\nu)$ are slowly varying functions.

4.1 Regularization

One way of making use of this additional assumption is a discretization of $\Delta u/u(x)$ and $F(\nu)$ in terms of known functions. In our method we use the cubic splines in the first case and the Legendre polynomials in the second case. The coefficients in these two representation together with $\Delta Y(1)$ are determined by least-squares method. A simple-minded application of this method results in a solution for $\Delta u/u(x)$ which exhibits artificial oscillations. There is a well-known cure to this problem known as regularization (e.g. Craig and Brown, 1986) which consist in additional term to minimized quantity.

We (DGPSa) considered various forms of regularization. The subject of the minimization was the quantity

$$\sum_{j=1}^{J}(\frac{\Delta\nu_j}{\sigma_j})^2 + \frac{\lambda}{J}\int[x^{ik}\frac{d^k}{dx^k}\frac{\Delta u}{f}]^2 dx, \tag{3}$$

where J is the total number of the frequency data and σ_j are errors. Values k, i and λ are control parameters for the regularization. We considered $k = 1$ or 2 and $i = 0$ or 1. The regularization parameter λ was chosen to be the minimum value that still suppressed the oscillations showed. Experiments were conducted in which attempt was made to reproduce known differences in $\Delta u/u$ between two solar models from differences in p-mode frequencies. The set of modes was the same as in the observational data with weights determined by the observational errors. The conclusion from these experiments was that for $x > 0.1$ it is possible to reproduce $\Delta u/u$ very accurately and independently of the choice of the type of the regularization. The results are also insensitive to λ in a wide range of values. However in the inner core the steep slopes like those seen in Fig. 5 for Model 2 and 4 could not be reproduced. A regularization needed do avoid the oscillatory behavior always flattened these real features. An alternative method must be used in the inner core. This will be described in subsections 4.3 and 4.4.

4.2 The Role of $F(\nu)$ and ΔY

If a modern standard solar model is used as the reference model then the $F(\nu)$ term is by far the dominant contributor to $\Delta\nu$. It is not our purpose to determine this function as we are interested here in probing the core and, in any case, interpretation of this function is not easy. It is, however, quite important for the present application to have a sufficient number terms (N_F) in the representation of the $F(\nu)$. Inadequate number of terms causes not only a poor fit but also a wrong determination of the sound speed in the core. This is illustrated in Fig.2 where against N_F plotted are χ, which is a measure of the fit, and $(\Delta u/u)_c$, the seismic correction to the model value of u in the center. The quantity χ is defined in a standard way,

$$\chi = \sqrt{\frac{1}{J - N_T}\sum_{j=1}^{J}\left(\frac{\Delta\nu_{\mathrm{res},j}}{\sigma_j}\right)^2}, \tag{4}$$

where N_T is the total number of free parameters. Here $N_T = N_F + 43$ (number of splines) + 1 ($\Delta Y(1)$)

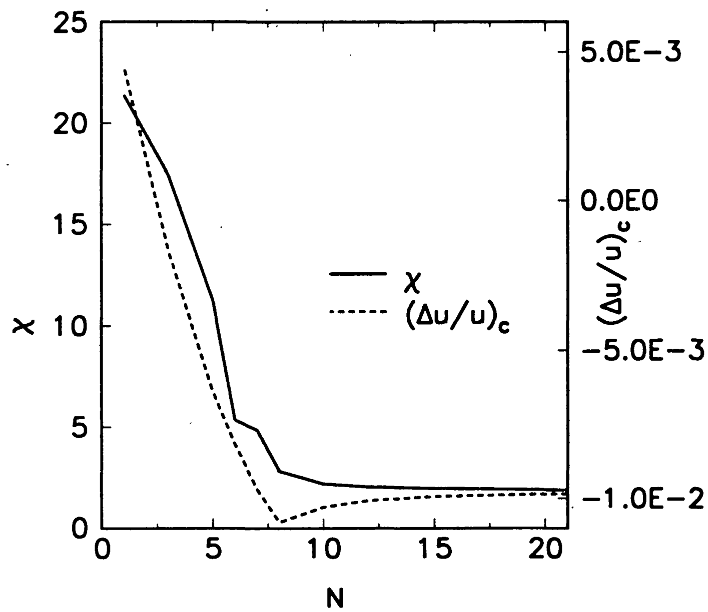

Figure 2 The ...ative seismic correction to the model value of u in the center and χ defined in Eq.(4) are plotted against number of Legendre polynomials used in representation of the $F(\nu)$ function.

If the gravitational He settling is not included in the reference model then allowing for the $Y(1)$ adjustment is also important for an accurate probing of the core. The difference in $\Delta u/u$ between inversions with and without this adjustment can be up to 5.0×10^{-3} near $x = 0.1$.

4.3 Optimal Averaging

In this method we abandon the idea of determining functions. Instead, we try to determine mean values weighted with (possibly narrow) kernels centered at selected x_0 values. We consider a linear superpositions of individual kernels for the modes present in the data set,

$$K_u(x_0, x) = \sum_j c_j(x_0)\mathcal{K}_{u,j}(x), \tag{5}$$

and determine coefficients c_j at selected points. An application of the classical method of Backus & Gilbert (1968) to helioseismic data was described by Gough & Thompson (1991). The SOLA method we adopted in our inversions (DGPSa, DGPSb) was developed by Pijpers & Thompson (1992). Here, one tries to construct kernels $K_u(x_0, x)$ which are as close as possible to Gaussians, G, characterized by their half-widths at half-maximum, w. Subject to least-square of minimization is the quantity

$$\int [K_u - G(\frac{x - x_0}{w})]^2 dx + \mu \sum_{j=1}^{J} c_j^2 \sigma_j^2. \tag{6}$$

The second term, with a trade-off parameter μ, is added to avoid large error magnification. For a specified w, larger μ leads to smaller errors in the localized mean but the kernels may differ significantly from Gaussians and therefore w cannot be regarded as the measure of localization.

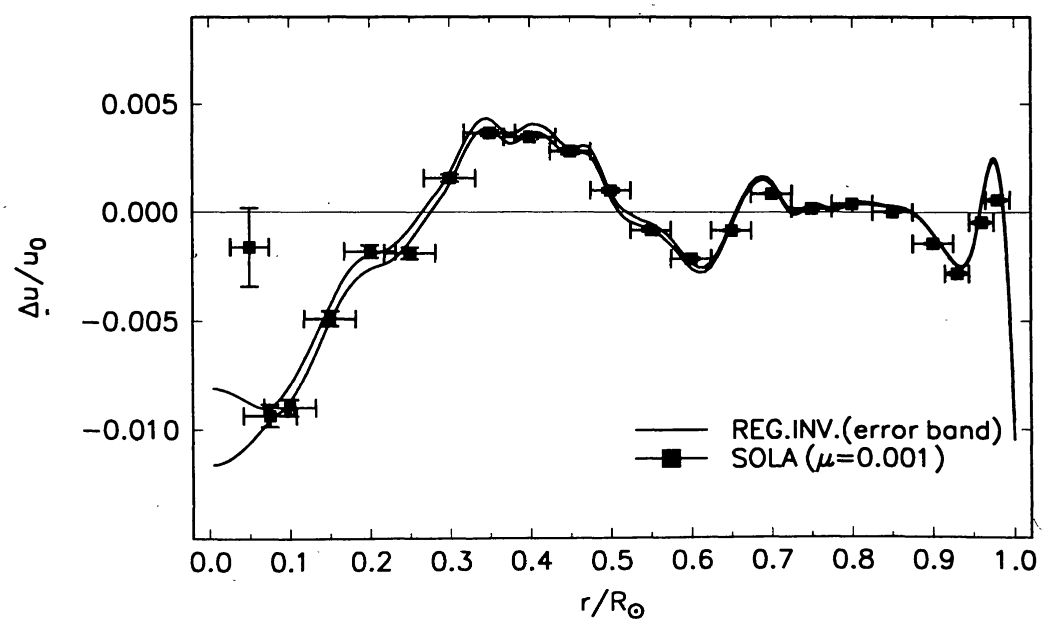

Figure 3. A comparison of the relative seismic corrections to the model function $u(x)$ obtained by two inversion methods. The bands for the regularized inversion were obtained by a numerical simulation of random errors in frequency data consistent with the σ values.

In Fig.3 results of the inversion by means of the two methods are compared. The reference model used here, which is the same as in DGPSa, is not completely up-to-date. In particular, it does not take into account elemental settling, but this is not relevant for the validity of inversion. Quoted values of $\Delta u/u$ are averages weighted with appropriate kernels and they are given by $\sum_j c_j \Delta\nu_{j,c}/\nu_j$, where $\Delta\nu_{j,c}$ are frequency differences with effects of ΔY and $F(\nu)$ removed. The errors of these values are given by $\sqrt{\sum_j c_j^2 \sigma_j^2}$. They are marked as vertical errors. The horizontal errors are given by w.

Everywhere in the model except for the inner core the agreement of the two methods is very good. At $x_0 = 0.05$ the SOLA value must be regarded as more realistic because regularization introduces an artificial smoothing. We should stress however the uncertainty in SOLA values is larger than the quoted errors. The errors shown in this figure reflect only the frequency errors quoted by the observers.

5. Seismic model of the solar core

We have seen that the two methods of inversion yield consistent results for $x > 0.1$. We thus may rely there on the functional form of $\Delta u(x)$ as obtained by means of the regularized least-square method. However, for a reliable and accurate probing of the inner core a different approach is required. In a hybrid method of seismic model construction developed in DGPSa we represent $\Delta u/u$ in the $[0, x_f]$ range as a truncated power series in x^2 thereby satisfying the boundary condition at $x = 0$. We make use of a few averaged values obtained with the SOLA method and the continuity conditions at x_f to determine the coefficients in this series.

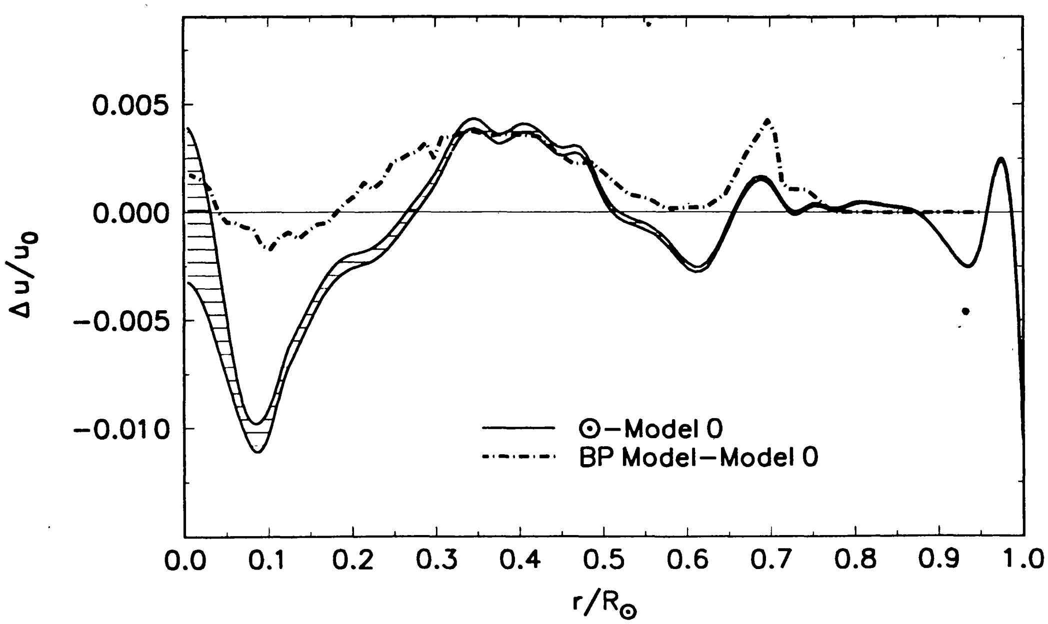

Figure 4. Relative seismic corrections to $u(x)$ relative to the reference model (Model 0) obtained by the hybrid method. The band is consistent with the frequency errors. The relative difference between the adopted reference model and the model calculated by Bahcall and Pinsonneault (BP) is also shown.

In Fig.4 the seismic corrections to our reference model obtained in this way are shown. In this case we adopted $x_f = 0.125$ and used the averaged SOLA values at $x_0 = 0.05$ and 0.1 calculated with $\mu = 0.01$. Note that, in this application, a good fit of the SOLA kernels to Gaussians is not important. We may see that $u(x)$ in the seismic model is very close to that in the standard model of Bahcall and Pinsonneault (1992, BP). This model is certainly superior to the reference Model 0 as it takes into account gravitational settling of helium and uses more reliable Fe abundance data in calculated opacities. In fact, these two differences tend to cancel each other in u. The superiority of the BP model is reflected primarily in the $Y(1)$ which is much closer to the seismic value.

The agreement between the seismic and the BP model in the central density is worse. The seismic value is 147.5 ± 0.8 while the BP value is 154.2 g cm^{-3}. These errors, as well as the error bands shown in Fig. 4, underestimate true uncertainty in central parameters. In particular, the choice $\mu = 0.001$ leads to a value for $\Delta u/u$ in the center which is by 1.5×10^{-2} larger than seen in Fig. 4. The difference, which may be a consequence of an inconsistency in the data, decreases rapidly with x.

Seismic corrections to various structural functions are evaluated in terms of $\Delta u/u$, with use of the integral formulae given in DGPSa. The seismic model that has been made available as a data file to be used for comparisons with theoretical models contains the error bands for u, ρ, P, M_r/M, $1/\Gamma_1$, and $d\log\rho/d\log P - 1/\Gamma_1$ are tabulated against x.

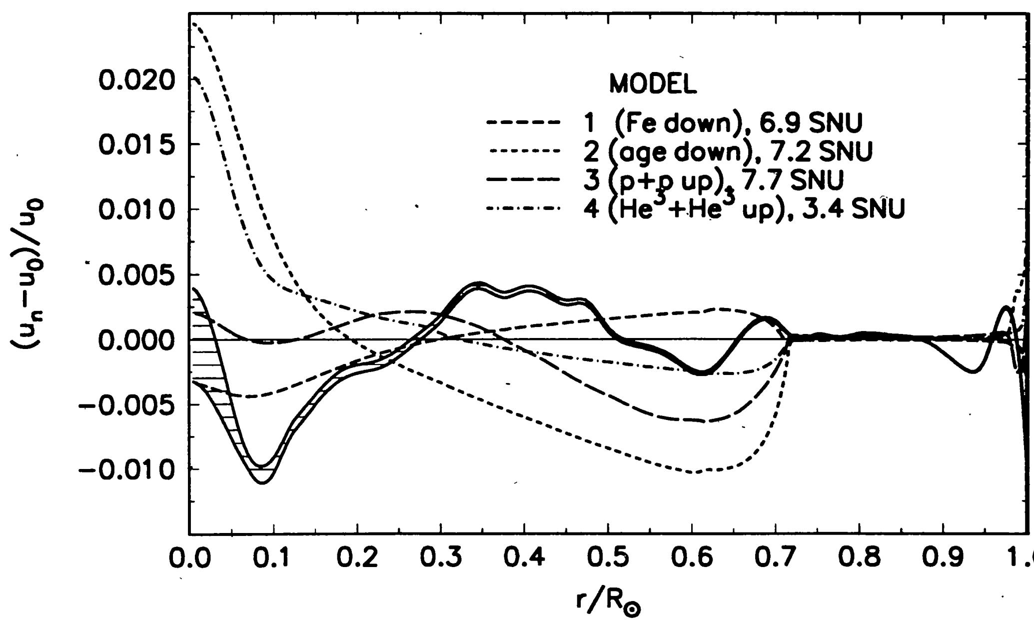

Figure 5. Relative differences in $u(x)$ between modified models and the standard reference model are compared with seismic correction. Model 1 has a lower Fe abundance (meteoritic value) implying lower opacity. Model 2 has an assumed age of 4 (instead of standard 4.6) billion yr. Model 3 has enhanced the p+p cross-section by a factor of 1.034 and Model 4 the He^3 +He^3 cross-section enhanced by a factor 9. Calculated neutrino counting rate for the Homestake experiments are provided for each of the modified models. The value for Model 0 is 8.2 SNU.

The good agreement between the seismic model and the modern theoretical model supports standard picture of solar evolution and may be used as an argument in favor of a nonastrophysical solution of the solar neutrino problem. The agreement is not perfect. The differences are significantly larger than the uncertainty implied by the observational errors. These are perhaps somewhat underestimated but certainly not to the extent needed to account for the Δu values. Particularly large differences are seen in the core near $x = 0.1$ It is important to identify the cause of these differences. Perhaps the improvements introduced recently by Bahcall and Pinsonneault (1996), which include taking into account the heavy element settling, improves the agreement.

In Fig. 5 $\Delta u/u$ inferred from the inversion is compared with the similar relative differences induced by various modifications in the input to solar models. It is important to notice that these modifications affect the u parameter in quite different ways. For Models 2 and 4 the largest effect occurs in the inner core. In this region, differences in u ($\propto T(1.25X + 0.745)$) reflect primarily differences in the hydrogen abundance, X and therefore differences in the past evolution. This is why a high accuracy in probing the solar core is crucial for testing stellar evolution theory.

A comparison of the differences in u between various models with the error band looks very encouraging. Though with current data this band underestimates the true uncertainty, there is no doubt that with forthcoming experiments even better precision will be reached.

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