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N-body problem

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"Sir Isaac established the rules, Poincaré presented the challanges".—Victor Szebehely

Abstract. This is a review article on the N-body problem. The development of the qualitative methods regarding the various topics of Celestial Mechanics are presented in a systematic way starting from the early researches of the particular solutions and the new integrals of motion to the recent development on the existence of periodic orbits, binary formation, escaping, KAM theorem and Chaos.

Key words : celestial mechanics—N-body problem

1. Introduction

The basic ideas of N-body problem were published in 1687 by Sir Isaac Newton in his Principia. The limitations to his work was given later by Henry Poincaré, who described the non-integrability principle as applicable to problems of three and more bodies.

The only known analytical solutions of the N-body problem are the Euler and Lagrange solutions that exist for any number of bodies and any mass ratios. Unfortunately these solutions are very exceptional. We can also obtain solutions from one another by rotations and by space-time translations. And the most interesting transformation is the scale transformation :

If $\bar{r}_i(t)$, $(i = 1, 2, \ldots, n)$ is a solution of the N-body problem then $k^2\bar{r}_i\ (t/k^3)$, $(i = 1, 2, \ldots, n)$ is another solution. In this case velocities are given by $k^{-1}\,\bar{v}_i\,(t/k^3)$, energy integral by h/k^2 and angular momentum by $k\bar{c}$.

The usual quantative methods, both analytical and numerical, give information about the dynamical system limited to the solution of interest and to a small vicinity. The accuracy also decreases and disappears as the time increases.

This drawback in quantitative methods has resulted in the progress of qualitative methods. Although they give partial information, yet they are valid for very long periods of time and generally for all time. We shall deal with these methods in section 6.

2. Statement of the N-body problem

To describe the motion of N-bodies which are moving under the Newtonian law of gravitation is called the N-body problem. In mathematical language it means given the position vector $\bar{x}_0$ and velocity vector $\dot{\bar{x}}_0$ at time t_0, to determine the position vector $\bar{x}$ and velocity vector $\dot{\bar{x}}$ at time t i.e.

$$\bar{x} = x_i\,(t,\, t_0,\, \bar{x}_0,\, \dot{\bar{x}}_0)$$

$$\dot{\bar{x}} = \dot{x}_i\,(t,\, t_0,\, \bar{x}_0,\, \dot{\bar{x}}_0) \qquad \text{... (2.1)}$$

It is quite natural to ask whether for any value of N or for any shape of bodies the problem has been solved or not. Our answer is both No and Yes. The answer is No because even for $N = 2$ we cannot describe the motion in the mathematical form (2.1). And the answer is Yes, because for $N = 2$ we agree that the problem has been solved when the bodies are either point-masses or spherical in shape though strictly not in the mathematical form (2.1). The well-known solution is (figure 1)

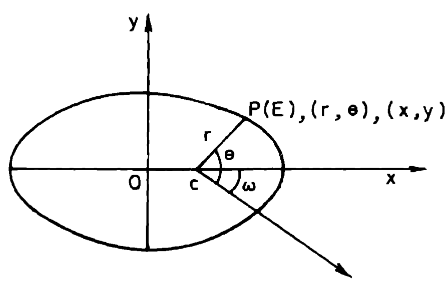

Figure 1. The two-body problem.

$$M = E - e \sin E, \quad M = n\,(t - \tau)$$

$$\theta = \omega + \cos^{-1}\left\{\frac{\cos E - e}{1 - e\cos E}\right\}$$

$$r = \frac{a\,(1 - e^2)}{1 + e\cos(\theta - \omega)}, \quad r^2\dot{\theta} = \left\{\mu a(1 - e^2)\right\}^{1/2},$$

$$x = a \cos E, \quad y = b \sin E,$$

$$\alpha_1 = -\frac{\mu}{2a}, \quad \beta_1 = -\tau$$

$$\alpha_2 = \sqrt{\mu a(1 - e^2)}, \quad \beta_2 = \omega.$$

3. Motion of rigid bodies

Major contributors to this problem are Duboshin (1958-1960), Kondurar (1963-69), Hor (1967), Holland (1969), Johnson (1969), Kinoshita (1970-1972), Hitzl (1971), Choudhry anc

Misra (1974), Gaida (1974), Barkin (1975-1979), Bhatnagar (1977-1986), El-Sabouri (1978), Mavraganis (1979). Duboshin (1958) has given the equations of motion in the form

$$m_i \ddot{\xi}_i = \frac{\partial u}{\partial \xi_i}, \; m_i \ddot{\eta}_i = \frac{\partial u}{\partial \eta_i}, \; m_i \ddot{\zeta}_i = \frac{\partial u}{\partial \zeta_i},$$

$$A_i \dot{p}_i - (B_i - C_i) q_i r_i = \left(\frac{\partial u}{\partial \psi_i} - \frac{\partial u}{\partial \phi_i}\cos\theta_i\right)\frac{\sin\phi_i}{\sin\theta_i} + \frac{\partial u}{\partial \theta_i}\cos\phi_i,$$

$$B_i \dot{q}_i - (C_i - A_i) r_i p_i = \left(\frac{\partial u}{\partial \psi_i} - \frac{\partial u}{\partial \phi_i}\cos\theta_i\right)\frac{\cos\phi_i}{\sin\theta_i} - \frac{\partial u}{\partial \theta_i}\sin\phi_i,$$

$$C_i \dot{r}_i - (A_i - B_i) p_i q_i = \frac{\partial u}{\partial \phi_i}$$

where

$$u = \frac{1}{2}\sum_{i,k}^{N} U_{ik}, \; i \neq k$$

$$U_{i,k} = f \int dm_i \int \frac{dm_k}{\Delta_{ik}},$$

$$p_i = \dot{\psi}_i \sin\phi_i \sin\theta_i + \dot{\theta}_i \cos\phi_i,$$

$$q_i = \dot{\psi}_i \cos\phi_i \sin\theta_i - \dot{\theta}_i \sin\phi_i,$$

$$r_i = \dot{\psi}_i \cos\theta_i + \dot{\phi}_i, \; i = 1, 2, .., N = (n+1)$$

The order of the above equations is 12 $(n + 1)$. These equations cannot be integrated as the translational and rotational motions are coupled together and in general they cannot be separated. However 10 classical integrals of motion do exist. In special cases more integrals may exist.

(i) Rigid bodies are spherical :

p_i, q_i, r_i = const., $i = 0, 1, 2, \ldots, n$ and u is independent of Euler's angles.

(ii) Rigid bodies are bodies of revolution

r_i = const., $i = 0, 1, 2, \ldots, n$ and u is independent of Euler's angle ϕ_i.

(iii) Δ_{ik}^{-3} etc. neglected in u; where Δ_{ik} is equal to the distance between any element of ith body and any element of kth body :

In this case translational and rotational motion are independent of each other.

(iv) Stationary Solutions : In the case of two rigid bodies a maximum of 36 stationary solutions exist (table 1).

(v) Satellite case : This case has been studied in depth by Duboshin (1960). In his paper, he has studied the problem of the motion of an artificial celestial body revolving

Table 1

S. No.	Nature of rigid bodies	No. of stationary solutions	Solutions
1.	Both triaxial (Bhatnagar 1986)	36	$L_i^j;\ i = 1, 2, ..., 6$; $j = 1, 2, ..., 6$
2.	Spheroid, triaxial (Bhatnagar 1980)	18	$L_i^j;\ i = 1, 3, 5$; $j = 1, 2, ..., 6$
3.	Both spheroid (Bhatnagar 1972)	09	$L_i^j;\ i, j = 1, 3, 5$
4.	Spherical, triaxial (Kinoshita 1972)	06	$L_i^j;\ i = 1$; $j = 1, 2, ..., 6$
5.	Sphere, spheroid (Kinoshita 1970)	03	$L_i^{j+1} = L_i^j = L_{i+1}^j\ \forall\ i = j$; $i, j = 1, 3, 5$

around a central planet and possessing rotational motion about its centre of inertia. Under such preassigned assumptions this problem admits simple particular solutions which serve practical interest. In these particular solutions, called regular motion, centre of inertia of the body moves around the planet along a circular Keplarian Orbit and the body preserves an invariable orientation relative to this orbit. Bhatnagar & Usha (1986) have studied this problem under category (B) and results are given in tables 1, 2 and 3 of their paper.

(vi) For the theory of the motion of several coupled rigid bodies about a fixed point, the reader is advised to consult the book by Eugene Leimanis (1965).

4. Three body problem $N = 3$

Major contributors are Euler and Lagrange (1772), Jacobi (1836), Hill (1878), Poincare (1892-99), Whittaker (1904), Levi-Civita (1904) and Birkhoff (1915). In recent times, a lot of work on three-body problem has been done by Marchal, Anosova, Hadjidemetriou, Zheng, Yoshida H. and Yoshida B., Maciejewski, Henrard, Froschle, Lemitre, Nobili, Bhatnagar, Sharma, Choudhry, Singh, Goldstein, Benest, Heggie, Brumburg, Valtonen, Orlov, Kiseleva Contopoulos, Aarseth, Alladin, Miller, Wisdom, Message, Kovalevsky, Jupp, Deprit, Garfinkal, Chazy, Giacaglia, Jefferys, Sundman etc. Chapters in books on Celestial Mechanics by Plummer (1918), Charlier (1907), Moulton (1914), Brouwer-Clemence (1961), Danby (1962), Mccuskey (1963), Pars (1965), Pollard (1966), Szebehely (1967), Tapley & Szebehely (1973) give lot of information about three-body problem. There is a comprehensive review paper on the three-body problem by Valtonen (1988).

4.1. *Classifications of motions*

Tapley & Szebehely (1973) have classified various types of motion and the same has been reviewed in detail by Bhatnagar (1990). We may state briefly as follows :

When $h >= 0$, $I = \infty$, $h = \frac{1}{2}\sum_{i=1}^{3} m_i v_i^2 - U$, $I = \sum_{i=1}^{3} m_i r_i^2$ the motion could be hyperbolic-explosion; hyperbolic or parabolic orbits and hyperbolic-elliptic (binary). When

$h < 0$ and I is bounded the motion could be either interplay or ejection or revolution or periodic. When $h < 0$ and $I \to \infty$, the motion could be hyperbolic-elliptic (binary) or parabolic-elliptic (binary).

We may also observe that escape orbits are dense; interplay leads to escape or ejection and repeated ejection leads to escape.

4.2. *Escape conditions*

In literature various conditions of escape or ejection are given. One can refer to these conditions in the book by Tapley & Szebehely (1973) or the paper by Bhatnagar (1990).

4.3. *Special solutions*

Following are some of the important results regarding 3-body problem :

(a) *Lagrang's solutions* : There are five well known Lagrang's solutions—three collinear L_i, (i = 1, 2, 3) and two triangular L_i (i = 4, 5).

(b) *Sundaman result* : It states that three-body bounded motions with a non-zero angular momentum cannot approach a triple collision.

(c) There are first 10 integrals; 6 from the motion of the centre of mass; 3 from angular momentum and 1 from energy.

(d) Lower bounds of the semi-moment of inertia

$$I = \frac{1}{2} \sum_{i=1}^{3} m_i r_i^2$$

can be given in terms of the integrals of motion.

(e) *Chazy conjectures* : Chazy has conjectured several hypothesis for the 3-body problem:

(i) For $h > 0, \bar{c} \neq 0, m_1, m_2, m_3 (\neq 0)$ there exists 49 possible combinations of original and final evolution. This is true even for large values of h and $\bar{c}$.

(ii) For $h < 0, \bar{c} \neq 0$ there are nine possible modes of evolution including two oscillatory types.

Chazy also conjuctured that $h < 0$ the motions of exchange type are impossible. The alternative to this conjecture is the existence of a complex cantor set structure for the

Table 2

Three-body problem—final evolutions

Class	Type	Conditions on $\bar{c}$ (angular momentum) or h (energy)*	Evolutions of R (= sup r_{ij}) and r (= inf r_{ij})
Singular type	Triple collision at t_c	$\bar{c} = 0$	Rand $r \sim (t_c - t)^{2/3}$
Hyperbolic expansions	Hyperbolic type	$h > 0$	$r \sim t$
	Hyper-parabolic type	$h > 0$	$R \sim t, r \sim t^{2/3}$
	Hyper-elliptic type	$h >< 0$	r bounded
Parabolic expansions	Tri-parabolic type	$h = 0$	$R \sim t^{2/3}, r \sim t^{2/3}$
	Para-elliptic type	$h < 0$	r bounded
Sub-parabolic	Bounded type	$h < 0$	$0 < b \leq r \leq R \leq B < \infty$ lim Sup $R = \infty$
Types	Oscillatory type I	$h < 0$	Lim Inf $R < \infty$, r bounded
	Oscillatory type II	$h < 0$	R bounded, Lt r inf $= 0$

exchange motions. The second Chazy conjecture was challenged by (Schmidt 1947; Khilme 1961; Alexeev 1956) and a numerical example presented by Alexeev remained inconclusive. However Szebehely (1975) computed a simple and symmetrical exchange motion for the masses $m_1 = 1$, $m_2 = m_3 = 2$. The second Chazy conjecture remains partially true. According to Marchal (1988) the final evolution of the three-body problem have been classified by Chazy in 1922 with some minor improvements (table 2).

5. Restricted three-body problem

Statement : Two bodies revolve around their centre of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two but not influencing their motion) moves in the plane defined by the two revolving bodies. The restricted problem of three bodies is to describe the motion of the third body.

Euler was the first to contribute towards the restricted problem in 1772 in connection with his Lunar Theories. His main contribution was the introduction of a synodic (rotating) co-ordinate system resulting in what is called the Jacobi integral which was discovered by Jacobi (1836). Implications of this integral are numerous. It determines the regions of motion. Its application to Celestial Mechanics was first made by Hill (1878). Poincare and Birkhoff are the pioneers in the qualitative methods of dynamics. Poincare's famous work in three volumes 'Methodes Nouvelles' completed in 1899 was so new and origional that many of its implications are still not clear. We give below some of the important results in regard to the restricted problem.

(i) *Stationary solutions* : There are five well known Lagrangian's solutions. Three collinear and two triangular

(ii) *Jacobi integral* : The problem has a well known Jacobi integral

$$v^2 = 2\Omega - c$$

where v is the speed of the infinitesimal mass and

$$\Omega = {}^1/_2\,(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

(iii) *Curves of zero-velocity* : (a) Curves of zero velocity are given by $2\Omega - c = 0$. They are shown in figure 2, for $c_1 > c_2 > c_3$. (b) When the infinitesimal mass moves in the vicinity of either the first primary or the second primary, it cannot escape and is said to have the 'Hill Stability'.

(iv) *Sundman inequality* : We know the inequality in the form

$$c^2 + \left(\frac{dI}{dt}\right)^2 \leq 4\,I\,(U + h)$$

(a) for $h \geq 0$, the inequality implies $IU^2 \geq -c^2h$

(b) for $h < 0$, the inequality implies successively

$$c^2 \leq 4I\,(U + h)$$

$$c^2h \geq 4Ih\,(U + h)$$

$$IU^2 + c^2h \geq I\,(U^2 + 4Uh + 4h^2) = I\,(U + 2h)^2 \geq 0.$$

These inequalities divide the zone of possible motion into three disconnected parts. Such disconnection never happens whenever $n > 3$. This is the major difference between the three body problem and more than three body problem.

(v) *Oscillatory motions* : Sitnikov (1960) has studied the oscillatory motions of type I in a symmetrical case of the restricted three-body problem. This study can be extended to general 3-body problem. This type of motion has a measure zero in phase-space, whereas the oscillatory motion of type II has a positive measure. Hadjidemetriou (1977) has studied oscillatory motion of type II.

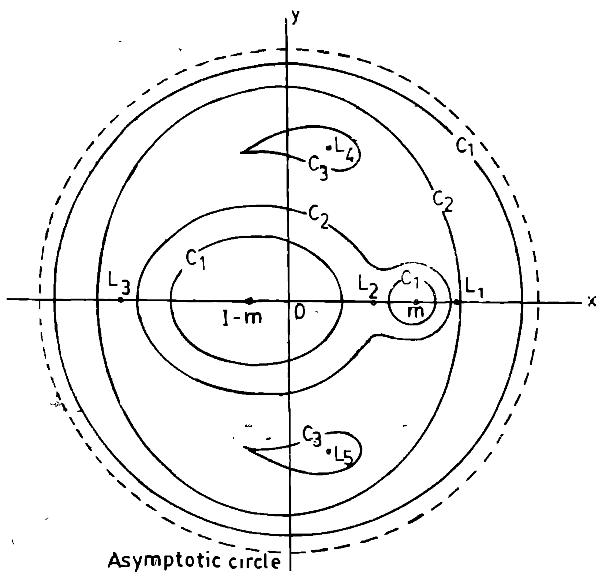

Figure 2. Curves of zero velocity in the restricted three-body problem.

6. For $n > = 4$

Major contributors are Elmabsout (1978, 1987, 1988), Bhatnagar and Monica (1986 a, b), Palmore (1979), Wintner (1941), Macmillan and Bartky (1932), William (1938), Giacaglia (1967).

(a) Elambsout (1988) has proved that for $n > = 4$, the configuration of relative equilibrium, where the bodies are at the vertices of a regular polygon with n sides, exists if and only if the masses are equal.

(b) Alexander Ollangren (1988) have studied the five-body problem under certain conditions. It is observed that there exists 9 stationary solutions. The topology of equipotential curves in the plane with 9 Lagrang's points is given in figure 3. They have observed that the central mass has a stabilizing effect on the motion of the 5th small body.

7. General N-body problem

As already stated, we can determine only 10 classical first integrals, 6 are due to the motion of the centre of mass, 3 from the angular momentum and 1 from energy. Mathematicians have been trying their best to find out new integrals but came out only with a negative theorem.

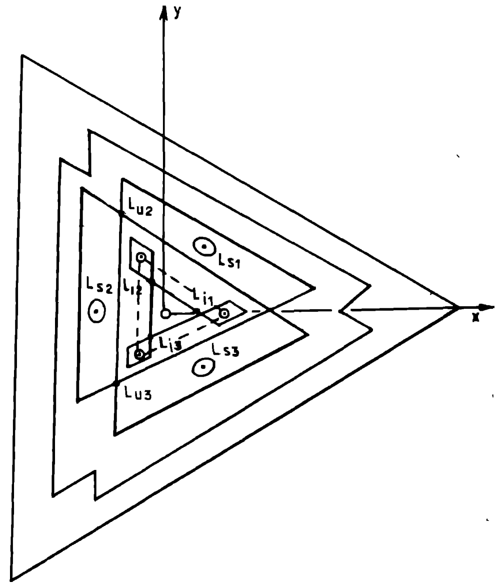

Figure 3. Topology of the equipotential curves in the xy-plane.

Theorem : No new integral of motion can be algebraic with respect to the velocities and arbitrary with respect to position. However partial integrals do exist. For $h > 0$, I (= semi moment of inertia) has one and only one minimum I_m along the solution of interest. I_m = const., exists along a solution. This is a true integral of motion. For $h < 0$ this becomes discontinuous.

Much information has been obtained about the N-body problem through the various qualitative methods. They give the general information and various other properties which are given below :

(i) *General information*

(a) Elambsout (1988) has shown that configuration of relative equilibrium where the N-bodies submitted to the Newtonian mutual attractions are at the vertices of a regular polygon with n sides, exists for $n >= 4$ if and only if the masses are equal.

(b) We have already stated Poincaré conjucture on periodic orbits, Chazy conjectures on original and final evolution and conjectures on the measure of oscillatory motions. Now we give some other main conjectures.

(c) The conjecture regarding Euler solutions. We know if h is not known, the bounded motion have still a lower bound of the semimoment of inertia 'I' very near to the limit of Euler motion.

Conjecture : The elliptic Euler motions give the limit, even in the case of unequal masses. We consider three given point masses and a given angular momentum. For the corresponding bounded motion the greatest lower bound of I is I_m and it is I_E for elliptic Euler motions. We know $I_E \geq I_m > 0.999 I_E$. The conjecture is that $I_m = I_E$ for all mass ratios and all angular momenta.

(d) *The conjectures on the structure of the set of solutions in phase-space* : (d.1) The periodic orbits and the tori of quasi-periodic orbits are significant for the set of solutions.

(d.2) Between these solutions, and for given values of the integrals of motion, the unbounded gaps are almost filled with open solutions. While the bounded gaps are almost everywhere filled with chaotic solutions dense in the gap.

(ii) *Classifications*

The classification of final evolution in the case of N-body problem is very complex. Table 3 gives one such classification by Marchel (1988).

Table 3

N-body problem—final evolutions

	Triple or multiple collision $R \to R_f : r \to 0$	
Singular types	Infinite expansion in a bounded interval of time	$R \to \infty;\ r \to 0$
	Super-hyperbolic expansion $R/t \to \infty;\ r \to 0$	
Hyperbolic expansions	All $\bar{r}_j = \bar{A}_j + O\ (t^{2/3})$; 2 to n subsystems; $R \sim t$	
	n-parabolic type $h = 0$; R and $r \sim t^{2/3}$	
Parabolic and sub-parabolic types	parabolic types, clusters, bounded types, oscillatory types etc...	$h < 0$ $R = O(t^{2/3})$ r bounded

The ergodic theorem (Halmos 1958; Marchal 1977) leads to another classification of the solutions.

(a) The open orbits go from infinity, in phase space, to infinity (hyperbolic or parabolic expansions, collisions, singular motions).

(b) The recurrent orbits have the Poisson stability property : They always come back an infinite number of times into any vicinity of any past or future state and have thus identical original and final evolutions without expansion to infinity.

(c) The abnormal orbits : These informations are of course insufficient for a complete determination of the relations between original and final evolutions of given solutions.

(iii) *Periodic orbits*

We all are familiar with the famous statement of Poincaré regarding periodic orbits—"*The periodic orbits are our best opportunity for the understanding of the three body problem*". This conjecture can be extended to N-body problem. Hadjidemetriou (1988) has presented an exhaustive review of periodic orbits which are of interest to Dynamical Astronomy and their relation to actual systems.

According to Marchel periodic orbits have many advantages :

(a) Their existence can be rigorously demonstrated.

(b) They can be computed with any given accuracy.

(c) The knowledge of one period gives the full knowledge of the solution.

(d) The families of periodic orbits are easy to follow step by step by the method of analytical continuation and they are connected to each other by the phenomenon of 'branching'.

(e) The unstable periodic orbits are the limit of asymptotic orbits while most of the first order stable periodic orbits are, according to the Kolmogorov-Arnold-Moser theorem, surrounded by families of tori of quasi-periodic orbits.

(f) The most interesting feature of periodic orbits remains the still undemonstrated "Poincaré conjucture" : the periodic orbits are certainly dense in the set of bounded orbits, they are thus a very efficient means of "exploration'.

Besides these according to Hadjidemetriou we have the following advantages :

(a) It is a very useful tool for the study of non-integrable dynamical system, because they determine critically the structure of phase-space.

(b) The study of periodic orbits which are close to actual motions plays an important role in the understanding of the general properties of such a system.

(c) The study of periodic orbits improves our knowledge as to why some resonant motions are stable and attract motion in their vicinity, and why some resonances are avoided (as in the case of the distribution of asteroids) ?

(d) The study of periodic orbit enables us to know more about the relations between resonance and unstability, non-integrability and unstability.

(e) Through the study of periodic orbits, one can understand the role of other forces (i.e. drag, tidal etc.) besides gravitational forces.

For a review of periodic orbits of the general N-body problem see Hadjidemetriou (1981, 1984). Other studies regarding periodic orbits may be seen in Hadjidemetriou (1975 a, b), Broucke (1975), Henon (1974).

Periodic orbits in barred galaxies have been also computed by Contopoulos and Papayanopoulos (1980), Contopoulos [1981 (b)], Papayanompoulos and Petrou (1983), Contopoulos (1980), Van Albada and Saunders (1983), Pfenniger (1984), Athanassoula *et al.* (1983) and Barbanis (1984). Periodic orbits with the characteristic of collision has been studied by Bhatnagar (1969, 1971, 1972). Szebehely's (1967) famous book gives a fairly good idea about the periodic orbits. Poincaré has shown the existence of three kinds of periodic orbits :

(a) $e = 0$, $i = 0$, e = eccentricity, i = inclination of the orbital plane.

(b) $e = 0$, $i \neq 0$.

(c) $e \neq 0$, $i \neq 0$.

(iv) *Singularities and regularization*

Major contributors to the problem of regularization are Thiele (1892), Painleve (1897), Levi-Civita (1903), Burrau (1906), Sundaman (1912), Plummer (1914), Birkhoff (1915), Murnaghan (1936), Szebehely (1952), Lemaitre (1955), Deprit and Delie (1962), Deprit (1963), Szebehely (1967), Bhatnagar (1972) and many others.

Whenever there is a collision of two particles the force acting between particles approaches infinity as the distance between particles approaches zero. Therefore at collision equations of motion show singularities. The regularization of the equations of motion, thus, becomes essential.

The regularization of the solution at collision can always be accomplished by introducing the eccentric anomaly since the collision of two bodies in any problem can be regularized in this way. Szebehely (1967) has dedicated a full chapter on regularization in his book 'Theory of Orbits'. In some problems two or more singularities are required to be regularized simultaneously. Transformations which regularize two or more singularities simultaneously are called global regularization. In the famous problem of Trojan asteroids, both Garfinkal (1970-74) and Bhatnagar and Beena (1993) have been able to regularize only one singularity and it is conjectured that the second singularity cannot be regularized simultaneously. In order to remove the singularity, new dependent and independent variables are introduced :

$$\xi = f(u), \; dt/d\tau = g\,(u).$$

The selection of f and g depends upon various considerations. The underlying principle is to slow down the phenomenon by stretching the time scale so that the approach of the actual velocity to infinity can be handled.

The Levi-Civita (1903) transformation is rarely used except in the restricted three-body problem. An implementation for the planar three-body problem has been described by Huang and Innanen (1983 b). Levi-Civita transformation of 2D has been extended to 3D by Bhatnagar (1972), but this has a limited application as ξ the third co-ordinate has been taken to the order of μ the mass parameter. The generalization of Levi-Civita transformation to 3D is not possible. This difficulty has been overcome by the famous KS transformation (1964) by a formulation in 4D which contains a redundancy condition. An example of a general two body regularization without co-ordinate transformation has been described independently by Burdet (1967) and Heggie (1973). Sharma (1984) has very successfully used KS transformation for studying the effects of Air-drag and oblateness of earth on the motion of a satellite. Bettis and Szebehely (1972) have studied some of the fundamental properties of the KS transformation. The KS transformation has also been used by Peter [1968(a), 1968(b)] in the general three-body problem. A more recent implementation of KS regularization for the general N-body problem employs an improved version of the FP4 integration method, including energy stabilization (Aarseth 1985).

Multiple regularization : Waldvogal (1972) has used global transformation for the planar problem with three non-zero mass points. The works of Zare and Szebehely (1975) and Alexander (1986) are noteworthy. The regularization method for the N-body problem is given by Heggie (1974). This beautiful formation uses KS regularization between each particle pair and requires a total of $4N\,(N-1)+1$ eqs. in the local centre of mass frame. The main advantage of global transformation is that no switching is needed and thus loss of accuracy is avoided. Aarseth (1988) has given a review on integration methods for small N-body systems ($N < 25$). Among the special methods described in some detail are two-body regularization and multiple regularization. Besides this, a new general purpose code for perturbed regularization is presented.

(v) *Stability*

We come across the question of stability in regard to properties of singular points as well as those of limit cycles for the former, the stability concerns the equalibrium; for the latter it relates to stationary motion on a limit cycle.

Definition : *Singular point*—Consider autonomous systems of two first-order equations

$$\dot{x} = P(x, y)$$

$$\dot{y} = Q(x, y)$$

A point (x_0, y_0) for which $P(x_0, y_0) = Q(x_0, y_0) = 0$ is called a singular point.
Definition : *Limit cycle*—The differential equations

$$\dot{x} = P(x, y), \dot{y} = Q(x, y)$$

admit occasionally special solutions represented by closed curves in the phase plane which we call limit cycles—a limit cycle is a closed trajectory such that no trajectory sufficiently near it is also closed (figure 4).

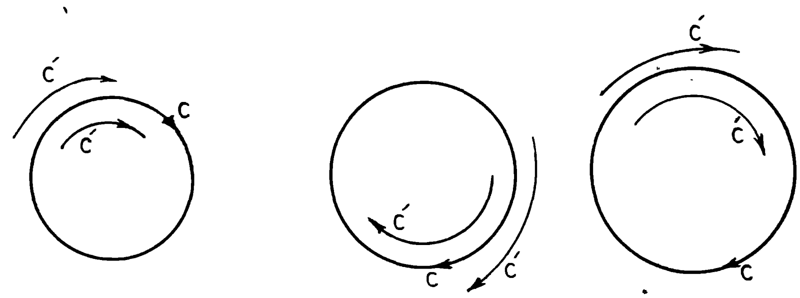

Figure 4. Limit cycles.

But in general we are interested in variational equations of Poincaré and direct method of Liapounov on stability. We now define most important types of stability:

(a) Suppose $U_1(t)$ is a solution of

$$\dot{x}_i = f_i(t, x_1, ..., x_n),\ i = 1, 2, ..., n,$$

$U_1(t)$ is said to be stable if, given $\varepsilon > 0$ *and* $t_0 \ \exists\ \eta = \eta(\varepsilon, t_0) \ni$ any solution $v_1(t)$ for which $|u_1(t_0) - v_0(t_0)| < \eta$ satisfies $|u_1(t) - v_i(t)| < \varepsilon\ \forall\, t > t_0$.

(b) *Asymptotic stability* : If $u_i(t)$ is stable and in addition $|u_i(t) - v_1(t)| \to 0$ as $t \to \infty$ then $u_1(t)$ is said to be asymptotically stable.

(c) *Orbital stability* : Suppose C is an orbit. C is said to be orbitally stable if given $\varepsilon > 0$, $\exists\ \eta > 0 \ni$ if R is a representative point of another trajectory which is within a distance η of C at time τ then R remains within a distance ε of C at time $t > \tau$.

(d) *Asymptotic orbital stability* : If C is orbitally stable and in addition the distance between R and C $\to 0$ as $t \to \infty$ it is said to be asymptotically orbitally stable. Orbital stability requires only that the orbits C and C' (closed trajectories) remain near each other whereas stability of motion (or of the solution) requires that in addition, the representative points R and R' (on C and C' respectively) should also remain close to each other. The analytic approach to the theory of stability is developed from the variational equations. Various methods have been developed to reduce variational

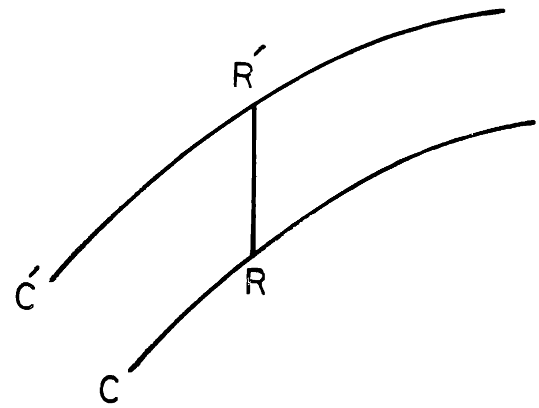

Figure 5. Orbital stability — stability of motion.

systems based on differential equation with periodic coefficients to those based on the differential equation with constant coefficients or which is the same, on the constant solution.

The variational equation of an autonomous system based on a constant solution is of the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\, x_j\,, \quad i = 1, 2, \cdots, n,$$

where a_{ij} are real constants. Its characterstic equation is

$$\begin{vmatrix} a_{11} - h & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - h & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - h \end{vmatrix} = 0.$$

we can discuss the stability of the system with the help of the roots of the above equation (called characterstic exponents).

If the characteristic exponents of the variational equation based on a constant solution $u_i(t) = \xi_i^0$ of an autonomous system have negative real parts, the solution is asymptotically stable. If at least one characteristic exponents has a positive real part, $u_i(t)$ is unstable.

However if variational equation based on a periodic solution is a linear system with periodic coefficients, the problem is more difficult. We can, sometimes, handle such a problem by applying a theorem due to Floquet. The details of this method are given by Minorsky (1969). The final result states that if the characteristic exponents of the variational equation based on a periodic solution have negative real parts, the solution is asymptotically stable. If one characteristic exponent has a positive real part, the solution is unstable.

The second method of Liapounov is based on properties of definiteness of certain functions associated with the differential system in such a manner that it is possible to ascertain whether the solution remains in a certain region or not. Four theorems on stability based on the direct method are discussed in detail by Minorsky (1969).

(vi) *Non-linear stability*

It has been observed many times that the pattern of stability changes drastically when we include non-linear terms also.

In some cases we can take a decision regarding non-linear stability on the basis of the variational equation consisting of linear terms only. Consider the non-linear autonomous system

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}\, x_j + X_i\left(x_1 \ldots, x_n\right), \quad (i = 1, 2, \ldots, n)$$

where p_{ij} are constants and X_i are power series of at least second degree. Then we can decide the non-linear stability of the point of equilibrium $x_i = 0$ from the characteristic equation corresponding to

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}\, x_j$$

If all roots λ_i of the characteristic equation have Re $(\lambda_i) < 0$, the point $x_i = 0$ is asymptotically stable whatever be the terms X_i.

However, if the characteristic equation does not have any root with positive real parts, but has some roots with zero real parts, then the terms in X_i can be chosen as to have either stability or instability. This is a critical case which requires specific investigation.

Non-linear stability of equilibrium points can also be decided by Arnold's Theorem (1961).

Theorem : If (i) $k_1\omega_1 + k_2\omega_2 \neq 0$ for all pairs (k_1, k_2) of integers and (ii) determinant $D \neq 0$ where ω_1, ω_2 are the basic frequencies for the linear dynamical system

$$D = \det (a_{ij}),\ i, j = 1, 2, 3$$

$$b_{ij} = \left(\frac{\partial^2 H}{\partial I_i \partial I_j}\right)_{I_i = I_j = 0}, \quad i, j = 1, 2$$

$$b_{i3} = b_{3i} = \left(\frac{\partial H}{\partial I_i}\right)_{I_i = 0}, \quad i, = 1, 2$$

$$b_{33} = 0$$

$$H = \omega_1 I_1 + \omega_2 I_2 + {}^1/_2\,(AI_1^2 + 2BI_1 I_2 + CI_2^2) + \ldots.$$

is the normalized Hamiltonian with I_1, I_2 as the action momenta co-ordinates, then on each energy manifold $H = h$ in the neighbourhood of equilibrium, there exist invariant tori of quasi-periodic motion besides the manifold and consequently the equilibrium is stable.

The above theorem has been applied successfully in deciding the non-linear stability of equilibrium points in some problems of Celestial Mechanics, Stellar Mechanics and Stellar

Dynamics by Deprit & Deprit (1967), Bhatnagar & Hallan (1983), Bhatnagar & Usha (1992) and Bhatnagar & Hallan (1992).

The chapter on stability in 'Lectures on Celestial Mechanics' by Siegel and Moser (1970) is extremely illuminating and no reader on stability can afford to miss it.

(vii) *Ergodic theory*

In general, ergodic theory is the study of transformations and flows from the point of view of recurrence properties, mixing properties, and other global dynamical properties connected with asymptotic behaviour.

Consider a Hamiltonian system

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \; \dot{p}_i = -\frac{\partial H}{\partial q_i}, (i = 1, 2, \ldots, n).$$

If we are given an initial condition and such equations can be uniquely solved then the corresponding solution gives us the entire history of the system.

Suppose x is a point in phase-space representing the system at a time t_0. Further suppose $T_t(x)$ denotes the point of the phase space representing the system at time $t + t_0$. From this we observe that T_t is a transformation of phase-space and, moreover, $T_0 = id$, $T_{t+s} = T_t \circ T_s$. Thus $(T_t : t \in R)$ is a one parameter or group of transformation of phase-space. In dynamics one is interested in the asymptotic properties of the family $\{T_t\}$.

Most of the work so far can be categorized into one of the four following types :

(a) *Measure theoretic* : Here one deals with a measure space X and a measure preserving transformation $T : X \to X$.

(b) *Topological* : Here X is a topological space and $T : X \to X$ is a continuous map.

(c) Mixture of (i) and (ii).

(d) *Smooth* : One considers a smooth manifold X and a smooth map.

The ergodic theorem (Halmos 1958, Marchal 1977) can be applied to the N-body problem and leads to another classification of the solution.

(a) The open orbits go from infinity, in phase space, to infinity (hyperbolic or parabolic expansions, collisions, singular motions).

(b) The recurrent orbits have the Poisson property : they always come back an infinite number of times into any vicinity of any past or future state and have thus identical, original and final evolutions without expansion to infinity.

(c) *The abnormal orbits* : These remaining orbits are infinitely rare (set of measure zero in phase space); they correspond to the 'complete capture', to the asymptotical motions, etc. The recurrent orbits are sometimes divided into bounded recurrent orbits (positions and velocities forever bounded) and oscillatory recurrent orbits. The informations are of course insufficient for a complete determination of the relations between original and final evolutions of given solutions.

(viii) *KAM Theorem*

We observe coherence and chaos in real systems. While Toda Lattice (one-dimensional chain of masses coupled by forces which vary exponentially with the separation of masses) is an example of complete coherence and a hard sphere gas complete chaos. Most systems lie somewhere in between and as we vary some parameter of the system, say energy parameter, both types of behaviour can be exhibited. This dual behaviour can be understood through KAM (Kolmogorov, Arnold, and Moser) work.

Suppose we have an integrable system with N degrees of freedom. Then its trajectories in the 2N-dimension phase-space lie on n-dimensional surfaces in the phase-space. These n-dimensional surfaces are called KAM surfaces. If we perturb such a system thereby making it non-integrable, the perturbation induces resonance zones locally in the phase-space which make the system chaotic in the region of the resonance zones. As the perturbation grows these resonance zones overlap and destroy KAM surfaces. When all KAM surfaces are destroyed, the trajectories are free to wander throughout the region of phase-space and the entire region becomes chaotic.

(ix) *Quasi-periodic*

In non-linear mechanics, it is possible in many cases to establish the existence of integral manifolds which have the property of asymptotic attraction of nearby trajectories. Consider the system

$$\dot{x} = X\,(x, \varepsilon) \qquad \ldots (7.1)$$

where $x = (x_1, x_2, \ldots\ldots, x_n)$, $X = (X_1, X_2, \ldots, X_n)$ are vectors of an n-dimensional Euclidean space and ε a small parameter. Under certain conditions it is possible to establish the existence of the invariant toroidal manifold

$$x = x\,(\phi),$$

$$\phi = (\phi_1, \phi_2, \ldots, \phi_n) \qquad \ldots (7.2)$$

for the system (7.1)

In that case the system (7.2) reduces to the following equations on the torus

$$\frac{d\phi}{dt} = \nu + f\,(\phi, \varepsilon) \qquad \ldots (7.3)$$

$\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ and $f\,(\phi, \varepsilon)$ is a periodic function. Under certain conditions the manifold (7.2) has the property of asymptotic attraction of the trajectories of any solution of equation (7.1) not lying on the torus.

The Poincaré-Denjoy theory deals with one dimensional case. Equation (7.1) becomes

$$\frac{d\phi}{dt} = \nu + f\,(\phi + \theta)$$

$$\frac{d\phi}{dt} = \omega.$$

According to this theory, the behaviour of the solution on a two-dimensional torus is characterized by the rotational number Ω. If (i) Ω is irrational, then the solution on the torus is quasi-periodic, and (ii) Ω is rational, then the solution is periodic. The rotational number Ω is the ratio of the basic frequencies ω_1 and ω_2 of the system (figure 6).

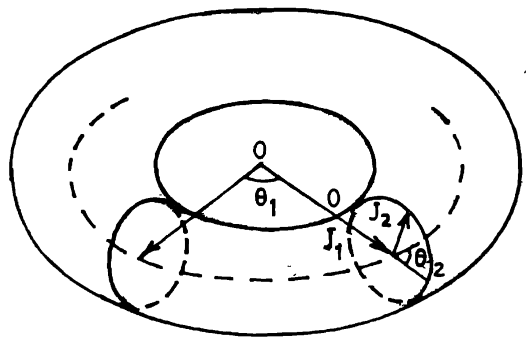

Figure 6. 2-Torus-quasi-periodic motion.

It has not been possible so far to investigate the general case of the manifold of higher dimensions, i.e. when the equations on the torus are reducible to equations of the form (7.3)

(x) *Integrals of motion*

Consider the system

$$\dot{x}_i = X_i\,(x_1, x_2, \ldots, x_n),\ i = 1, 2, \ldots, n, \text{ which has the solution}$$

$$x_i = u_i\,(t).$$

We define integral of motion by the differentiable function $F(x_1, x_2, \ldots, x_n)$ on a domain D of the phase-space, and not constant on any open set such that

$$F(u_1(t), u_2(t), \ldots, u_n(t)) = \text{const.}$$

Integrals of motion give us lot of information about the system even though we may fail to obtain all the integrals of motion. In the case of restricted three-body problem, we can specify regions of motion through Jacobi integral.

(xi) *Existence and uniqueness*

The properties of existence and uniqueness play a very important role in understanding various dynamical systems.

The theorem of Cauchy regarding the existence and uniqueness of a solution of a differential equation is well known. It not only guarantees the existenc of solutions with prescribed initial conditions, but it asserts that the initial conditions determine the solution uniquely, even though we may not be able to solve the differential equation.

We use, in general, any of the following four methods for showing the existence of periodic orbits :

(a) Method of analytic continuation,

(b) Process of Fourier coefficients,

(c) Application of fixed point theorem,
(d) Method of power series.

These methods are discussed in detail by Szebehely (1967).

(xii) *Chaos*

In the past thirty years classical mechanics has undergone a tremendous change. New ideas and techniques have been developed to understand the mechanism by which systems can undergo a transition from regular to chaotic behaviour.

The phenomenon of chaos was discussed by Poincaré, and re-discovered by Lorenz and Henon with modern computers. In literature we find this phenomenon with different names : (a) Erratic motion, (b) semiergodic or completely ergodic motion, (c) stochastic motion, and (d) random orbits.

Here we are concerned only with deterministic systems leading to undeterministic phenomena.

Definition : Almost all bounded motion with at least one positive Liapunov characteristic exponent (LCE) is chaotic:

$$\mathrm{LCE} = \underset{t\to\infty}{\mathrm{Lt}} \frac{1}{t} \log |\mu_j(t)|, \ \mu_j = \text{eigenvalues}.$$

Suppose there is a dynamical system

$$\frac{dX}{dt} = AX + O(X).$$

We have the following important results :

(a) All LCE < 0, $\Rightarrow x(t) \to$ lt. point
(b) One LCE, = 0, others < 0 $\Rightarrow x(t) \to$ lt. cycle.
(c) pLCE = 0, others < 0 $\Rightarrow x(t) \to$ lt. p torus.
(d) One or several LCE > 0 $\Rightarrow x(t) \to$ strange attractor

The last is the dissipative version of chaotic motion. Now, we discuss some predictive criteria or conditions for chaos.

(a) *Non-linear systems* : A chaotic system must have non-linear elements or properties. A linear system cannot lead to chaos.

(b) *Random input* : A small change in the initial conditions may change regular motion to chaotic motion.

(c) *Phase-space* : Chaotic motions have orbits in phase-space which never close or repeat.

(d) *Fourier spectrum* : Initially there is a dominant frequency component ω_0. If subharmonics frequency spectrum ω_0/n appears, Chaos is likely to appear (figure 7).

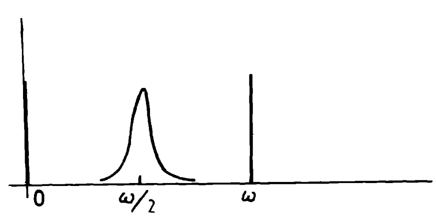

Figure 7. Fourier spectrum of chaotic motion.

(e) *Poincaré map* : *Definition* : If $x_n \equiv x(t_n)$, $y_n \equiv \dot{x}(t_n)$, then the points (x_n, y_n) is a Poincare map if t_n is choosen according to certain rule (figure 8).

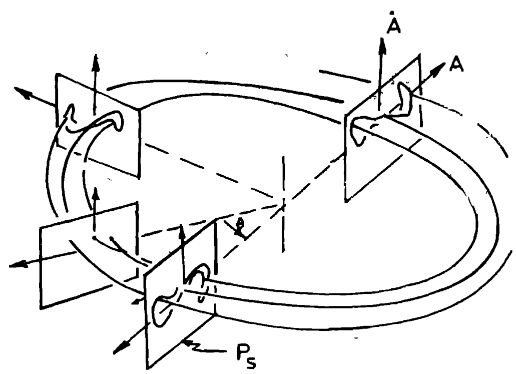

Figure 8. Poincaré map.

If the Poincare map does not consist of either a finite set of points or a closed orbit then the motion may be chaotic. The appearance of fractal patterns in the Poincare map is a strong indicator of chaotic motion.

(f) *Period doubling* : We start with a periodic motion. Then as we vary an experimental parameter, λ, say, the motion undergoes a bifurcation or change to a periodic motion the period of which is double that of the previous motion. As we change λ further, again we come across another bifurcation or change to another periodic motion the period of which is again doubled. Feigenbaum (1978-82) discovered a marvellous result. He showed that the critical values of λ at which successive period doubling occurs obey the following rule :

$$\underset{n \to \infty}{\mathrm{Lt}} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta = 4.6692016$$

This process accumulates at a critical value of λ after which the motion becomes chaotic (figure 9).

(g) *Quasi-periodic to chaos* : Suppose ω_1 and ω_2 are the basic frequencies of the motion. We know that (i) if ω_1/ω_2 = rational, the motion is periodic, (ii) if ω_1/ω_2 = irrational, the motion is quasi-periodic. Such motions are imagined to take place on the surface of a torus where Poincaré map represents a plane which cuts the torus. We often get chaotic motions when quasi-periodic torus structure is broken up as the system parameter is broken up.

(h) *Intermittancy* : It has been observed that sometimes there are long periods of periodic motion with bursts of chaos. This phenomenon is called intermittancy. As we vary the experimental parameter the chaotic bursts are more frequent and longer.

(i) *KAM surface* : We have seen earlier that when all KAM surfaces are destroyed the trajectories are free to wander throughout the regions of phase-space and the entire region becomes chaotic.

(j) *Lyapunov exponents* : Chaos in deterministic systems implies a sensitive dependence on initial conditions. This means if two trajectories start close to each other, they will move exponentially away from each other after sometime. Through this

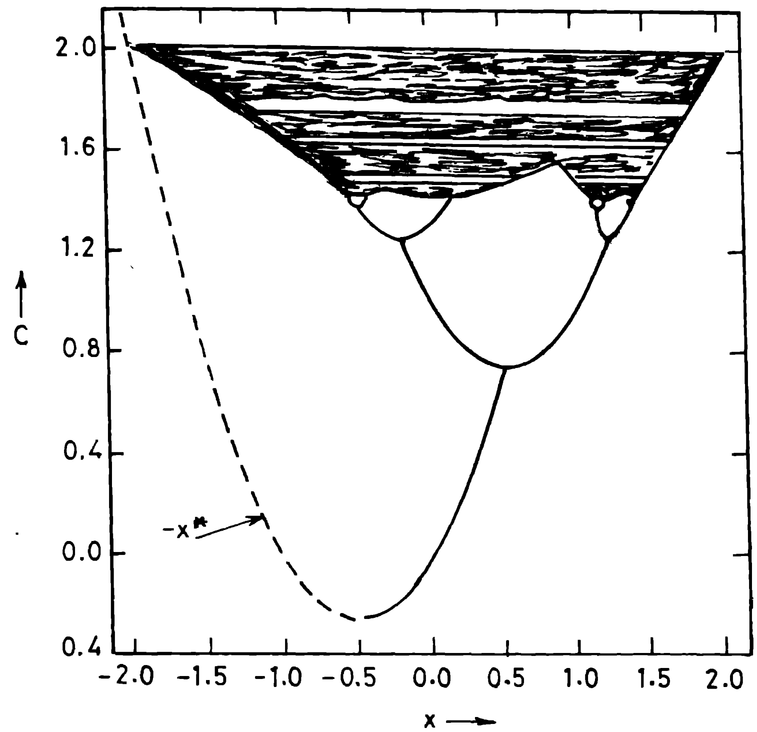

Figure 9. Bifurcation diagram for equation $X_n + 1 = X^2$ from Grebogi, Ott & Yorke (1982f).

property a criterion for chaos has been developed in terms of Lyapunov exponents (LE) (Wolf *et al.* 1985). This review by Wolf *et al.* contains two computer programs for calculating LE.

We define LE

$$\lambda = \frac{1}{t_N - t_0} \sum_{k=1}^{N} \log_2 \frac{d(t_k)}{d_0\,(t_{k-1})}$$

where d_0 measures the initial distance between two starting points and d their distance at time t. If $\lambda > 0$ the motion is chaotic. If $\lambda < 0$ the motion is regular. Schuster (1985) has calculated λ in a given example.

(k) *Some examples in celestial mechanics and stellar dynamics* : (i) Ignoring small tidal effects, motion of a planet seems to be quasi-periodic type. (ii) Motions of galaxies and star clusters are of open type and they are continuously loosing some stars. (iii) Motion of a comet is either open or quasi-periodic or temporary chaotic. (iv) Rotation of Hyperion is chaotic. (v) Kirk-Wood gap may be due to chaotic motion.

(l) *Picture in phase-space* : In conservative and analytic problems, the general phase-space picture is as follows :

(i) Reduction is achieved by integral of motion.

(ii) After utmost reduction, we have either periodic or quasi-periodic solutions.

(iii) Between these periodic or quasi-periodic solutions the holes of infinite measure are filled with open solutions while the holes of finite measure are filled with chaotic solutions which are dense in the holes.

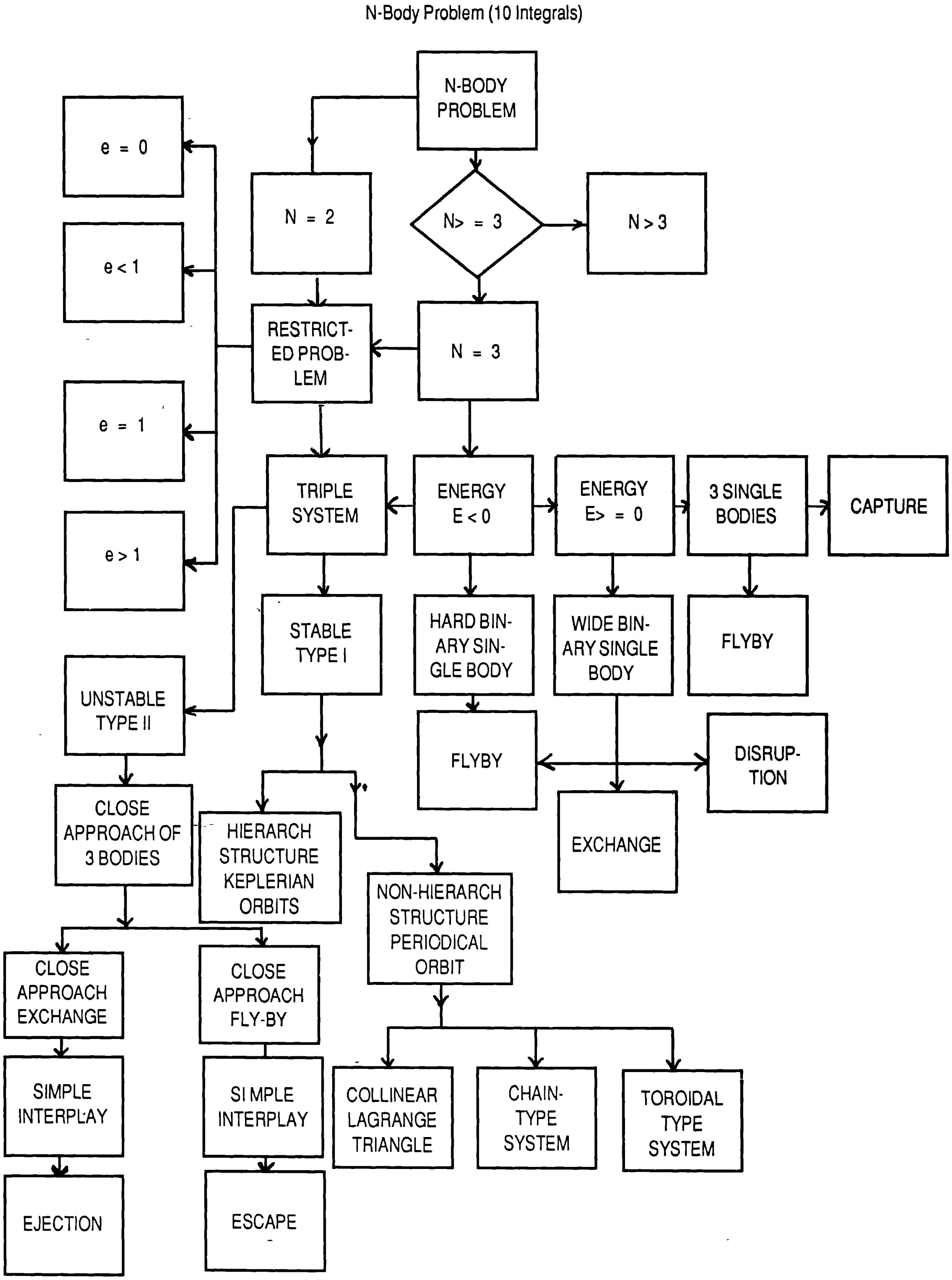

Figure 10. The types of motion in the N-body problem.

(iv) Temporary chaotic motion is extremely developed in nearly closed holes.
(v) Statistical methods give excellent results when applied to chaotic motions.

Flow chart of N-body problem

Figure 10 shows the flow chart of the N-body problem according to Anosova (1988). This gives the types of motion in the gravitational N-body problem.

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